Title: S.B.s' $\dot{R}$- Sampling Distribution and its Application in Hypothesis Testing

AUTHOR: SURAJIT BHATTACHARYYA*
Department of Mathematics, S.A. Jaipuria College (The University of Calcutta), Kolkata, India

***Corresponding author**: Surajit Bhattacharyya.

ORCID ID: 0000 – 0001 – 6293 – 9775

Abstract:

Hypothesis testing empowers individuals and organizations to make smarter, more confident decisions by relying on evidence rather than assumptions. It provides the foundation for drawing meaningful conclusions from data by promoting fair evaluations of new ideas or performance metrics. Testing a statistical hypothesis is a two-action decision problem after the experimental sample values have been obtained,with the two actions being acceptance or rejection of the theory underconsideration.

The entire large sample theory is based on applyingthe 'normal test'. However, if the sample size n is very tiny, the distributions of various statistics are far from normality In such cases, exact sample tests, pioneered by W.S. Gosset and later developed and extended by Prof.R. A. Fisher, are used.

The F-test following Snedecor's F-distribution is an exact sample test; however, it has several limitations. By convention, whenpreparing the test,the greater of the two variancesmaybe taken in the numerator, and the smaller one will be in the denominator to ensure the F-ratio ≥ 1 under H_1 andto increase the sensitivity to make it easier to compare with right-tailed critical values.

To eliminate this limitation, a new $\dot{R}$- distribution ($\dot{R}$- test) has beenproposed in this paper. In this test,there is no such ambiguity.All important properties and the comparisons with other tests have been discussed in detail.

Mathematics subject classification (2020): 62E15, 62G10, 60E05, 60E99, 97K80

Keywords: Sampling distribution, Testing of Hypotheses, Critical Region, $\dot{R}$-Sampling Distribution, $\dot{R}$ -Constant, Significant Values.

1. Introduction:

1.1. Background:

In statistical investigations, our ultimate interestgenerally lies in one or more characters possessed by the members of the population. By taking a sample, we observe the form or values of the characters for the individuals included in the sample. Suppose, that there is only one character of importance (e.g.x), and if x_i is the value of x for the *ith* member of the sample, then$x_1, x_2, \ldots, x_n$ are the sample observations. Any statistical measure calculated on the basis ofsample observations is called a statistic, e.g., the sample mean, sample standard deviation, proportion of defects, etc.

Again, one of the primary interests will be to know the values of different measures of the population distribution of x, such as its mean, and standard deviation. A measure of this type, which is calculated on the basis of population values of x, is called a parameter. The object of sampling is to study the features of the population on the basis of sample observations. A carefully selected sample is expected to reveal these features and hence we infer the population from a statistical analysis of the sample.

In almost all sampling techniques, the composition of samples depends only on chance. Therefore, if many samples of a fixed size are drawn from a given population, the group of units constituting the sample also varies from one sample to another. These differences in the values of a statistic are called sampling fluctuations. The parameter has no sampling fluctuations. The frequency distribution of the statistic that would be obtained if the number of samples of the same size (e.g. n) were infinite is called the sampling distribution of the statistic.

The mean of a statistic will generally be the corresponding parameter, exactly or approximately. The standard error i.e. the standard deviation of the statistic, then gives an idea of the average error that one would commit in using the statistic instead of the parameter.(Knowledge of the sampling distribution is necessary in finding confidence limits for parameters and in testing statistical hypotheses.)

There are two types of problems. First, we may have no information about some population characteristics, especially the values of the parameters involved in the distribution, and we want to obtain estimates of these parameters. This is the problem of estimation. Second, some information or hypothetical values of the parameters may be available. It is necessary to test how far the hypothesis is tenable in light of the information provided by the sample. This is the problem of the test of hypothesis or test of significance.

An essential aspect of sampling theory is the study of the tests of significance, which enable us to decide on the basis of sample results, if

(i) The deviation between the observed sample statistic and the hypothetical parameter value, or (ii) The deviation between two sample statistics, is significant or might be attributed to chance or, sampling fluctuation. Let$x_1, x_2,\dots, x_n$ be a random sample from a population of a known mathematical form that involves an unknown parameter θ. I should try to find two functions t_1 and t_2 on the basis of sample observations such that the probability of θ being included in the interval(t_1,t_2) has a given value say e

$$P\ (t_1<\theta <t_2\ ) = e$$

This interval is called a confidence interval for θ. The two quantities $t_1$and $t_2$are known as confidence limits.

The significance of confidence limits is that if many independent random samples are drawn from the same population and the confidence interval is calculated for each sample, then the parameter will be included in the intervals in the$e$ proportion of cases in the long run. Thus the estimate of the parameter is stated as an interval with a specified degree of freedom.

In the tests of significance, we start with a certain hypothesis about the population characteristics. This is called the null hypothesis and is denoted by the symbol H_0 . For example, the null hypothesis may be that the population mean is μ_0 , we may write,

$$H_0\colon \mu=\mu_0$$

Any hypothesisthat differs from the null hypothesis is called the alternative hypothesis,which is denoted by the symbol H_1 and is any one of the following:

$$H_1\colon \mu \neq\mu_0\ ;\ H_1\colon \mu>\mu_0;\quad H_1\colon \mu<\mu_0$$

The sample is then analyzed to decide whether to reject or not reject the null hypothesis H_0.

Remarks: At the time of formulation of a testing problem devising a ‘test of hypothesis’ the roles of H_0 and H_1 are not symmetric. To decide which of the two hypotheses should be taken as the null hypothesis and which one the alternative hypothesis, the intrinsic difference between the roles and the implications of these two terms should be clearly understood. The consequences of wrongly rejecting a null hypothesis seem more severe than wrongly accepting it.

In this paper, I discuss only the testing of the hypothesis with the help of a new sampling distribution

1.2. Looking back:

The chi-square distribution was first described by the German statistician Friedrich Robert Helmert (1875) who computed the sampling distribution of the sample variance of a normal population.Later on, this distribution was independently rediscovered by the English mathematician Karl Pearson in the context of goodness of fit, published in 1900, with a computed table of values published in (Elderton 1902). The idea of a family of "chi-square distributions", however, is not due to Pearson but arose as a further development due to Fisher in the 1920s. In their paper(1983) R. L. Plackett and Karl Pearson [12] discussed details about Chi-square distribution. Furthermore, we know about the chi-square distribution from the works of H.O. Lanchester [8], [9] in 1966 and 1969.

The *t* distribution was first derived as a posterior distributionin 1876 by Helmertand Lüroth. The *t* distribution also appeared in a more general form as a Pearson type IV distribution in Karl Pearson's 1895 paper. This distribution takes its name from William Sealy Gosset's 1908 paper, which defined his t in a slightly different way and investigated its sampling distribution, somewhat empirically, in a paper entitled 'The Probable Error of Mean'published in 1908. In 1926, Prof. R.A.Fisher [2], [3], [4], later, defined his own't' andprovided a rigorous proof of its sampling distribution. The salient feature of't' is that the statistic and its sampling distribution are functionally independent of σ, the population standard deviation. Although the distribution of 't' is asymptotically normal for large n, it is far from normal for small samples. The student t ushered in an era of exact sample distributions and tests. Since its discovery, many important contributions have been made toward the development and extension of small (exact) sample theory.
F distribution, also known as the Fisher–Snedecor distribution, was developed by the British statistician Sir Ronald A. Fisher in 1928. Snedecor tabulated the distribution in 1934, using the letter F in honor of Fisher. The *F* statistic is a ratio (a fraction) with two sets of degrees of freedom; one for the numerator and another for the denominator.The *F* distribution is a continuous probability distribution that arises frequently in statistical analyses, particularly when comparing variances.

Neyman & Pearson [11] considered a different problem (which they called "hypothesis testing" by taking two simple hypotheses). They calculated two probabilities and typically selected the hypothesis associated with the higher probability (the hypothesis more likely to have generated the sample). Their method always selects a hypothesis.

An approximation for the chi-square integral by Gray, H.L, Thompson, R.W. and Mc – Williams, G.V. (1969)[7], an approximation of thestudent's *t* distribution by Gentleman, M. W and Jenkins, M. A. (1968)[5], and an approximation of the F- distribution by George, E. O. and Singh, K. P. (1987)[6] need to be mentioned with great respect.

1.3. Looking at Present:

The "Adaptive estimation of a quadratic function by model selection" by Laurent, B., Hunter(2000)[10] and, Probability for Statisticians by Shoraek, G. R. (2000)[13] are of high importanceand very recent works. Some researchers [1] have discussed Statistical theoriesin the Fuzzy field.

1.4. Motivation and Novelty:

Today, the F distribution remains an important tool in statistical analysis and is widely used in many areas of research and industry. Its ability to compare variances and test hypotheses makes it useful in various applications.

However, this distribution has several limitations. One of them is thatthe greater of the two variances is to be taken in the numerator andthat thesmaller one will be in the denominator to increase the sensitivity.The decision may be different if two variances are interchanged. Toovercome this problem and harassment, I propose a new distribution (Test) in this article.

The test introduced in this paper is based onclassical information theory, which places computations in a context that has become well-developed and understood in the literature. The majority of its properties have been developed and proven in detail.

1.5. Looking ahead:

I hope that the new distribution will be widely used today in many fields, including psychology, medicine, engineering control systems, tax transformation, machine learning, and AI. It will also be useful in statistical tests for the equality of variances in the context of experimental design.More research work might be worked outto enhance the idea,and that is very important also in the field of fuzzy mathematics.

1.6. Structure of the paper:

Section2.1defines d.f, and sections 2.2, 2.3, 2.4, 2.5, and 2.6 describethe normal, chi-square, F-distributions, Bartlett's and Levene's test respectively. In Section3, a new sampling distribution is introduced and explained including a figure of its graphs. Its various properties are discussed in Section 4 with its relations with t and F-distribution. Section 5 presents some assumptions to bemade for the $\dot{R}$- test, Section 6 discusses its applications, Section 6.1 depicts how to perform the $\dot{R}$- test for the equality of population variances, and Section 6.2 discusses critical values of the $\dot{R}$- distribution with a clear diagram and three examples. Section 7presentsdetailed comparisons with F-distribution, 8 consists of a simulation report and 9 indicates the concluding remarks. Finally, the paper ends with the appendix (10) which consists of two tables regarding significant points of the $\dot{R}$ distribution.

2. Preliminaries:

2.1. Definition: Degree of freedom: (d.f.)

The number of independent variables that make up the statistic is the degree of freedom (d.f.). The number of degrees of freedom, in general, is the total number of observations less than the number of independent constraints imposed on the observations. For example, if k is the number of independent constraints in a set of nobservations then d.f = (n- k).

In addition, if any of the population parameters are calculated from the given data and used for computing the expected frequencies then in applying the χ^2- test of goodness of fit, we have to subtract one d.f. for each parameter calculated. Thus if 's' is the number of population parameters estimated from the sample observations(n), then the required d.f. for the χ^2- test is (n – s -- 1).

2.2. Definition: Normal distribution

The normal distribution is a continuousprobability distributiondefined by the density function (p.d.f)

$$f(x) = \frac{1}{\sigma\ 2\pi} e^{\frac{-(x-\mu)^2}{2\sigma^2}}\ ;\ (-\infty < x < \infty).$$

Where μ = mean, σ = standard deviation [π and e are two mathematical constants]

If a random variable x is normally distributed with mean μ and standard deviationσ, then $z = \frac{(x-\mu)}{\sigma}$is called the standard normal variable. It has the density function

$$f(z) = \frac{1}{2\pi} e^{\frac{-z^2}{2}}\ ;\ (-\infty < z < \infty).$$

This is a special case of a normal distribution with a mean of 0 and a standard deviation of 1.

2.3. Definition: Chi-square distribution.

A random variable x is said to follow chi-square (χ^2) distribution if its p.d.f. is of the

$$f(x) = K.e^{-x/2} x^{\left(\frac{n}{2}\right)-1}\ ;\ (0 < x < \infty).$$

Where $K = \frac{1}{2^{n/2}\Gamma(n/2)}$ is a constant [Γ is the gamma function]

The parameter n (positive integer) is called the number of d.f. Here mean = n and$S.D. = \sqrt{2n}$

In general, if, X_i, (i =1, 2, …, n) are independent normal variates with mean μ_i and variances σ_i^2 (i =1, 2, …, n), then

$\chi^2 = \sum_{i=1}^{n}(\frac{X_i - \mu_i}{\sigma_i})^2$ is a chi-square variate with n d.f.

2.4. Definition: Snedecor Fdistribution

A random variable$x$ is said to follow an F distribution with d.f.$n_1$andn_2 if its p.d.f. is of the form $f(F) = K \cdot x^{\frac{n_1}{2}-1}(n_2 + n_1 x)^{-(n_1+n_2)/2}$; $(0 < F < \infty)$.

Where $K = \frac{(\frac{n_1}{n_2})^{\frac{n_1}{2}}}{B(\frac{n_1}{2},\frac{n_2}{2})}$ is a constant [B (a, b) = Beta function]

In general, if ${\chi_1}^2$ and ${\chi_2}^2$ are two independent chi-square variates with $n_1$andn_2 degree of freedom (d.f.) respectively, then the$F$- Statistic is defined as follows.

$$F = \frac{\frac{{\chi_1}^2}{n_1}}{\frac{{\chi_2}^2}{n_2}}.$$

2.5. Definition: Bartlett'sDistribution

The test statistic follows a chi-squareddistributionwith k−1 degrees of freedom under the null hypothesis.

Chi-Squared PDF:

$f(x;\nu) = \frac{x^{\nu/2} e^{x/2}}{2^{\nu/2}\Gamma(\nu/2)}$, $x > 0$Where:

- ν is the degrees of freedom
- Γ is the gamma function

Test statistics: To testwhether multiple groups have equal variances.

Null Hypothesis is $H_0 = \sigma_1^2 = \sigma_2^2 = - - - - - - = \sigma_k^2$

$\chi^2 = \frac{(N-k)\ln(s_p^2) - \sum_{i=1}^{k}(n_i - 1)\ln(s_i^2)}{1 + \frac{1}{3(k-1)}\left(\sum_{i=1}^{k}\frac{1}{n_i - 1} - \frac{1}{N-k}\right)}$.Where

- $N = \sum_{i=1}^{k} n_i$
- s_i^2 = sample variance of group i.
- k = number of groups.
- s_p^2 = pooled variance.
- n_i = sample size of group$i$.

2.6. Definition: Levene'sDistribution

The test statistic follows an F-distribution with$d_1 = k - 1$ and $d_2 = N - k$ degrees of freedom.

Distribution PDF:

$$f(x; d_1, d_2) = \frac{\sqrt{\left(\frac{d_1 x}{d_1 x + d_2}\right)^{d_1}\left(\frac{d_2}{d_1 x + d_2}\right)^{d_2}}}{xB\left(\frac{d_1}{2}, \frac{d_2}{2}\right)} \quad where \ x > 0 \text{ and}$$

- d_1 = numerator degrees of freedom
- d_2 = denominator degrees of freedom
- B(a,b) = Beta function

Test statistics: Tests whether multiple groups have equal variances, but are more robust to non-normality. It is an ANOVA on absolute deviations from the group means (medians).

Null Hypothesis is $H_0 = \sigma_1^2 = \sigma_2^2 = ------ = \sigma_k^2$.

$Z_{ij} = |Y_{ij} - \overline{Y}_i|$. Where

- $\overline{Y}_i$ is the group mean or median.

3. <u>Definition</u>: SBs' $\dot{R}$ sampling distribution:

If $\chi_1{}^2$ and $\chi_2{}^2$ are two independent chi-square variates with n_1 and n_2 degrees of freedom (d.f) respectively, then the $\dot{R}$- Statistic is defined as

$$\dot{R} = \frac{\frac{\chi_1{}^2}{n_1+n_2}}{\frac{\chi_2{}^2}{n_2}}.$$

In other words, $\dot{R}$ is defined as the ratio of two independent chi-square variates, in the numerator the first variate is divided by the sum of two d.f. Whereas in the denominator the second variate is divided by the corresponding d.f. and follows the $\dot{R}$ **-** distribution with (n_1, n_2) d.f. with a probability function given by ---

$$f(\dot{R}) = \frac{\left({(n_1+n_2)}/{n_2}\right)^{\frac{n_1}{2}}}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \ \mathbf{x} \ \frac{\dot{R}^{\frac{n_1}{2}-1}}{\left[1+\left(\frac{n_1+n_2}{n_2}\right)\dot{R}\right]^{n_1+n_2/2}} \quad , \quad 0 \le \dot{R} < \infty.$$

<u>Derivation of the **SB s'** $\dot{R}$ distribution</u>: [Graphs of the $\dot{R}$-distribution for several d.f. are described in Fig.(01)].
Since $\chi_1{}^2$ and $\chi_2{}^2$ are two independent chi-square variables with n_1 and n_2 (d.f) respectively, their joint probability differential is given by ---

$$dF(\chi_1{}^2, \chi_2{}^2) = \frac{1}{2^{\frac{n_1+n_2}{2}}\Gamma(\frac{n_1}{2})\,\Gamma(\frac{n_2}{2})} \exp\left\{-\frac{(\chi_1{}^2+\chi_2{}^2)}{2}\right\} \mathbf{x} \ (\chi_1{}^2)^{({n_1}/{2})-1}(\chi_2{}^2)^{({n_2}/{2})-1} \, d\chi_1{}^2 d\chi_2{}^2.$$

Let us perform the following transformation of the variables –

$$\dot{R} = \frac{\frac{\chi_1{}^2}{n_1+n_2}}{\frac{\chi_2{}^2}{n_2}},\ u = \chi_2{}^2,\ 0 \le \dot{R} < \infty \text{ and } 0 < u < \infty.$$

Therefore, $\chi_1{}^2 = \frac{n_1+n_2}{n_2}\dot{R}\,.\chi_2{}^2 = \frac{n_1+n_2}{n_2}\dot{R}\,\mathrm{u}$ $and\ \chi_2{}^2 = u.$

The Jacobian of transformation ***J*** is given by ---

$$\boldsymbol{J} = \frac{\partial(\chi_1{}^2,\chi_2{}^2)}{\partial(\dot{R},\mathrm{u})} = \begin{vmatrix} \frac{\partial\chi_1{}^2}{\partial\dot{R}} & \frac{\partial\chi_2{}^2}{\partial\dot{R}} \\ \frac{\partial\chi_1{}^2}{\partial u} & \frac{\partial\chi_2{}^2}{\partial u} \end{vmatrix} = \begin{vmatrix} \frac{n_1+n_2}{n_2}\mathrm{u} & 0 \\ \frac{n_1+n_2}{n_2}\dot{R} & 1 \end{vmatrix} = \frac{n_1+n_2}{n_2}\mathrm{u}\,.$$

Thus, the distribution of the transformed variable —

$$dG(\dot{R},\mathrm{u}) = \frac{1}{2^{\frac{n_1+n_2}{2}}\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})}\exp\left\{-\frac{\mathrm{u}}{2}\left(1+\frac{n_1+n_2}{n_2}\dot{R}\right)\right\}\mathbf{x}\left(\frac{n_1+n_2}{n_2}\dot{R}\mathrm{u}\right)^{({n_1}/{2})-1}(\mathrm{u})^{({n_2}/{2})-1}|\,J\,|du\,d\dot{R}$$

$$= \frac{\left(\frac{n_1+n_2}{n_2}\right)^{{n_1}/{2}}}{2^{\frac{n_1+n_2}{2}}\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})}\exp\left\{-\frac{\mathrm{u}}{2}\left(1+\frac{n_1+n_2}{n_2}\dot{R}\right)\right\}(\mathrm{u})^{({n_1+n_2}/{2})-1}\dot{R}^{({n_1}/{2})-1}\,du\,d\dot{R}\,.$$

$$where \quad 0 \le \dot{R} < \infty \text{ and} 0 < u < \infty.$$

Integrating w.r.t u over the range 0 ---∞**,** the distributionof -$\dot{R}$ becomes

$$f\,(\dot{R})d\dot{R} = \frac{\left(\frac{n_1+n_2}{n_2}\right)^{{n_1}/{2}}\dot{R}^{({n_1}/{2})-1}d\dot{R}}{2^{\frac{n_1+n_2}{2}}\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})}\ \left[\int_0^\infty \exp\left\{-\frac{\mathrm{u}}{2}\left(1+\frac{n_1+n_2}{n_2}\dot{R}\right)\right\}(\mathrm{u})^{\left(\frac{n_1+n_2}{2}\right)-1}du\right]$$

$$= \frac{\left(\frac{n_1+n_2}{n_2}\right)^{{n_1}/{2}}\dot{R}^{({n_1}/{2})-1}}{2^{\frac{n_1+n_2}{2}}\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})}\frac{\Gamma(\frac{n_1+n_2}{2})}{\left[\frac{1}{2}\left(1+\frac{n_1+n_2}{n_2}\dot{R}\right)\right]^{\frac{n_1+n_2}{2}}}d\dot{R}.$$

$$f(\dot{R}) = \frac{\left({n_1+n_2}/{n_2}\right)^{\frac{n_1}{2}}}{\mathrm{B}\left(\frac{n_1}{2},\frac{n_2}{2}\right)}\ \mathbf{x}\ \frac{\dot{R}^{\frac{n_1}{2}-1}}{\left[1+\left(\frac{n_1+n_2}{n_2}\right)\dot{R}\right]^{({n_1+n_2})/{2}}}\quad,\quad 0 \le \dot{R} < \infty.$$

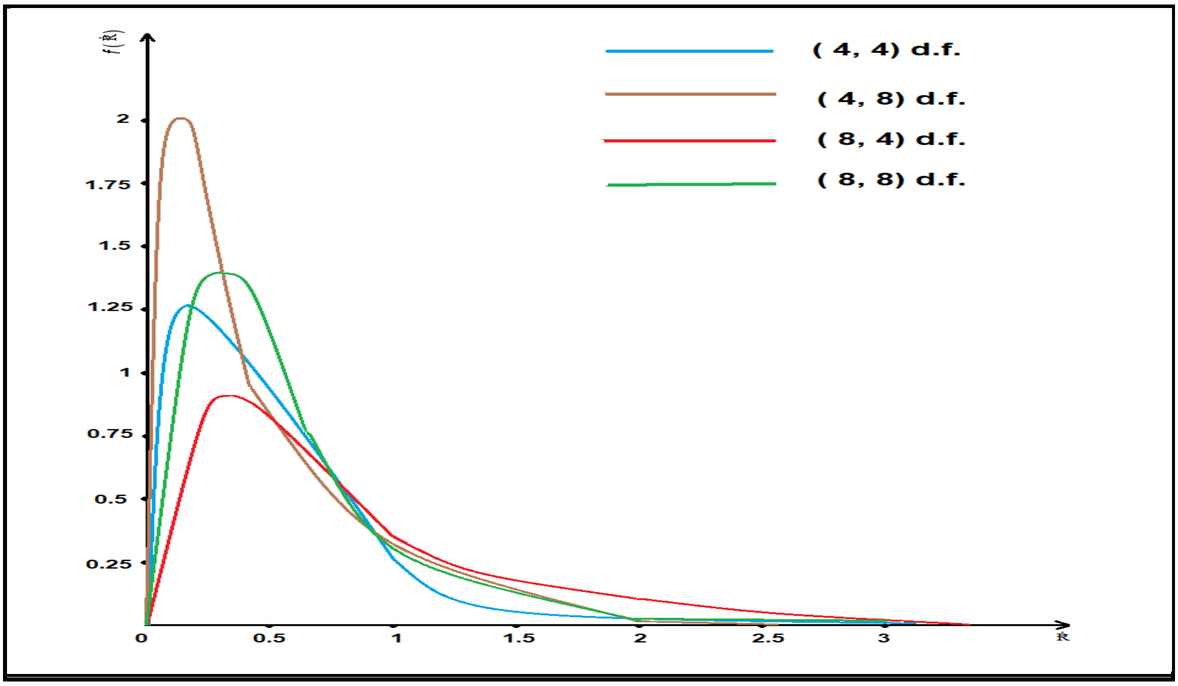

Figure: 01. Graphs of -Ṙ- distribution curves for various degrees of freedom.

The probability density function of $\dot{R}$ is as follows:

$$f(x) = \begin{cases} c x^{\frac{n_1}{2}-1}\left[1+\left(\frac{n_1+n_2}{n_2}\right)x\right]^{-{(n_1+n_2)}/{2}} & if\ x\epsilon\,[0,\infty\,) \\ 0 & if\ x\ elsewhere \end{cases} \text{ where } c = \frac{\left({n_1+n_2}/{n_2}\right)^{\frac{n_1}{2}}}{B\left(\frac{n_1}{2},\frac{n_2}{2}\right)}$$

Proof: $\int_0^\infty f(x)dx = c\int_0^\infty x^{\frac{n_1}{2}-1}\left[1+\left(\frac{n_1+n_2}{n_2}\right)x\right]^{-{(n_1+n_2)}/{2}} dx$

$$= c\ \frac{n_2}{n_1+n_2}\int_0^\infty \left(\frac{n_2}{n_1+n_2}\ t\ \right)^{\frac{n_1}{2}-1}(1+t)^{-\left(\frac{n_1+n_2}{2}\right)} dt \quad \left[\text{ taking}\left(\frac{n_1+n_2}{n_2}\right)x = t\ \right].$$

$$= c\ \left(\frac{n_2}{n_1+n_2}\right)^{\frac{n_1}{2}}\int_0^\infty (\ t\ )^{\frac{n_1}{2}-1}(1+t)^{-\left(\frac{n_1+n_2}{2}\right)} dt$$

$= c\ \left(\frac{n_2}{n_1+n_2}\right)^{\frac{n_1}{2}} B\left(\frac{n_1}{2},\frac{n_2}{2}\right) = 1$ [By setting the value of c].

4. **Properties** of the Ṙdistribution:

4.1. Expectation of the Ṙdistribution:

$$E(x) = \int_0^\infty x\, f(x)dx = c\int_0^\infty x.x^{\frac{n_1}{2}-1}\left[1+\left(\frac{n_1+n_2}{n_2}\right)x\right]^{-(n_1+n_2)/2} dx$$

$$= c\int_0^\infty x^{\frac{n_1}{2}}\left[1+\left(\frac{n_1+n_2}{n_2}\right)x\right]^{-(n_1+n_2)/2} dx. \quad [Here, \quad c = \frac{\left(n_1+n_2/n_2\right)^{\frac{n_1}{2}}}{B\left(\frac{n_1}{2},\frac{n_2}{2}\right)}]$$

$$= c\frac{n_2}{n_1+n_2}\int_0^\infty \left(\frac{n_2}{n_1+n_2}\,t\right)^{\frac{n_1}{2}}(1+t)^{-(n_1+n_2)/2}\,dt, \qquad [\text{ taking } \left(\frac{n_1+n_2}{n_2}\right)x = t\,]$$

$$= c\left(\frac{n_2}{n_1+n_2}\right)^{\frac{n_1}{2}+1}\int_0^\infty t^{\left(\frac{n_1}{2}+1\right)-1}(1+t)^{-[(\frac{n_1}{2}+1)+(\frac{n_1}{2}-1)]}\,dt$$

$$= c\left(\frac{n_2}{n_1+n_2}\right)^{\frac{n_1}{2}+1} B\left(\frac{n_1}{2}+1,\frac{n_2}{2}-1\right)$$

$$= \frac{\left(n_1+n_2/n_2\right)^{\frac{n_1}{2}}}{B\left(\frac{n_1}{2},\frac{n_2}{2}\right)}\ \mathbf{x}\ \left(\frac{n_2}{n_1+n_2}\right)^{\frac{n_1}{2}+1}\ \mathbf{x}\ B\left(\frac{n_1}{2}+1,\frac{n_2}{2}-1\right), \quad [\text{By putting the value of } c\ ].$$

$$= \left(\frac{n_2}{n_1+n_2}\right)\frac{\Gamma(\frac{n_1}{2}+\frac{n_2}{2})}{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})}\frac{\Gamma(\frac{n_1}{2}+1)\Gamma(\frac{n_2}{2}-1)}{\Gamma(\frac{n_1}{2}+1+\frac{n_2}{2}-1)} = \frac{n_2}{n_1+n_2}\frac{\Gamma(\frac{n_1}{2}+1)\Gamma(\frac{n_2}{2}-1)}{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})}$$

$$= \frac{n_2}{n_1+n_2}\frac{n_1}{2}\frac{1}{\frac{n_2}{2}-1} = \frac{n_1 n_2}{(n_1+n_2)(n_2-2)},$$

$where\ \frac{n_2}{2}-1>0$, or, $\frac{n_2}{2}>1$, or, $n_2>2$. [Because both the arguments of the beta function must be strictly positive]

4.2. Variance of $\dot{R}$ distribution:

Variance = $E(x^2) - [E(x)]^2$

$$E(x^2) = \int_0^\infty x^2 f(x)dx = c\int_0^\infty x^2.x^{\frac{n_1}{2}-1}\left[1+\left(\frac{n_1+n_2}{n_2}\right)x\right]^{-(n_1+n_2)/2} dx$$

$$= c\int_0^\infty x^{\frac{n_1}{2}+1}\left[1+\left(\frac{n_1+n_2}{n_2}\right)x\right]^{-(n_1+n_2)/2} dx. \qquad [Here, \quad c = \frac{\left(\frac{n_1+n_2}{n_2}\right)^{\frac{n_1}{2}}}{B\left(\frac{n_1}{2},\frac{n_2}{2}\right)}]$$

$$= c\frac{n_2}{n_1 + n_2}\int_0^{\infty}\left(\frac{n_2}{n_1 + n_2}\, t\right)^{\frac{n_1}{2}+1}(1+t)^{-\,{(n_1+n_2)}/{2}}\, dt, \qquad \left[\text{taking } \left(\frac{n_1 + n_2}{n_2}\right)x = t\right]$$

$$= c\left(\frac{n_2}{n_1 + n_2}\right)^{\frac{n_1}{2}+2}\int_0^{\infty} t^{\left(\frac{n_1}{2}+2\right)-1}(1+t)^{-\left[\left(\frac{n_1}{2}+2\right)+\left(\frac{n_1}{2}-2\right)\right]}\, dt$$

$$= \frac{\left(\frac{n_1+n_2}{n_2}\right)^{\frac{n_1}{2}}\left(\frac{n_2}{n_1+n_2}\, t\right)^{\frac{n_1}{2}+2}}{B\left(\frac{n_1}{2},\frac{n_2}{2}\right)} \mathbf{x}\ B\left(\frac{n_1}{2}+2, \frac{n_2}{2}-2\right)$$

$$= \left(\frac{n_2}{n_1 + n_2}\right)^2 \mathbf{X}\ \frac{\Gamma\left(\frac{n_1}{2}+\frac{n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} = \left(\frac{n_2}{n_1 + n_2}\right)^2 \frac{\Gamma\left(\frac{n_1}{2}+\frac{n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \frac{\Gamma\left(\frac{n_1}{2}\right)\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}-1\right)\left(\frac{n_1}{2}+1\right)}{\Gamma\left(\frac{n_1}{2}+\frac{n_2}{2}\right)\left(\frac{n_2}{2}-2\right)}$$

$$= \left(\frac{n_2}{n_1 + n_2}\right)^2 \frac{\left(\frac{n_1}{2}\right)\left(\frac{n_1}{2}+1\right)\Gamma\left(\frac{n_2}{2}\right)}{\Gamma\left(\frac{n_2}{2}\right)\left(\frac{n_2}{2}-2\right)\left(\frac{n_2}{2}-1\right)} = \left(\frac{n_2}{n_1 + n_2}\right)^2 \frac{(n_1+2)n_1}{(n_2-4)(n_2-2)}$$

Therefore, the required variance $= \left(\frac{n_2}{n_1+n_2}\right)^2 \frac{(n_1+2)n_1}{(n_2-4)(n_2-2)} - \left[\frac{n_1 n_2}{(n_1+n_2)(n_2-2)}\right]^2,$

$$where\ n_2 > 4$$

4.3. Mode of the $\dot{R}$ distribution**:**

We know that $f(\dot{R}) = c\frac{\dot{R}^{\frac{n_1}{2}-1}}{\left[1+\left(\frac{n_1+n_2}{n_2}\right)\dot{R}\right]^{(n_1+n_2)/2}}$ where $c = \frac{\left({n_1+n_2}/{n_2}\right)^{\frac{n_1}{2}}}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)}$ is a constant.

Therefore, $\log f(\dot{R}) = logc + \left\{\frac{n_1}{2}-1\right\}\log(\dot{R}) - \frac{n_1+n_2}{2}\log\left[1+\left(\frac{n_1+n_2}{n_2}\right)\dot{R}\right]$

$$\frac{\partial \log f(\dot{R})}{\partial \dot{R}} = \left(\frac{n_1}{2}-1\right)\frac{1}{\dot{R}} - \frac{n_1 + n_2}{2}\frac{1}{1+\left(\frac{n_1+n_2}{n_2}\right)\dot{R}}\frac{n_1 + n_2}{n_2}.$$

$$\frac{\partial f(\dot{R})}{\partial \dot{R}} = 0 \quad \Rightarrow \quad \frac{n_1 - 2}{2\dot{R}} - \frac{(n_1 + n_2)^2}{2[n_2 + (n_1 + n_2)\dot{R}} = 0$$

$$\Rightarrow\ \dot{R} = \frac{n_2(n_1 - 2)}{(n_1 + n_2)(n_2 + 2)}.\ \ where\ n_1 > 2.$$

It can be easily verified that at this point,-$f''(\dot{R}) < 0$. Hence, Mode = $\frac{n_2(n_1-2)}{(n_1+n_2)(n_2+2)}$.

Note:

i) Since, $\dot{R} > 0$, the mode exists iff$n_1 > 2$.

ii) Since Mode = $\frac{n_2}{(n_1+n_2)}\frac{(n_1-2)}{(n_2+2)}$, the mode of the $\dot{R}$ - distribution is always less than unity.

iii) The Mode is at 0 when$n_1 \leq 2$. This is because the $\dot{R}$- Distribution starts at zero and has no maximum value when $n_1 \leq 2$.

When $n_1 > 2$, the mode will shift as the degrees of freedom change.

iv) Karl Pearson's coefficient of skewness is given by $S_k = \frac{Mean-Mode}{\sigma} > 0$,

Since, in this case, Mean > Mode. Hence the $\dot{R}$ - distribution is positively skewed.

v) The probability p ($\dot{R}$) increases steadily at first until it reaches its peak (corresponding to the modal value which is less than 1) and then decreases slowly to become tangential at $\dot{R} = \infty$i.e. the $\dot{R}$- axis isasymptotic to the right tail.

4.4. Higher moments:

The kth moment of the $\dot{R}$ random variable X is well-defined as

Proof: From the definition of moments we obtain ------

$$\mu_x(k) = E[X^k] = \int_{-\infty}^{\infty} x^k\, f\,(x) = c\int_0^{\infty} x^k . x^{\frac{n_1}{2}-1}\left[1+\left(\frac{n_1+n_2}{n_2}\right)x\right]^{-\,{(n_1+n_2)}/{2}} dx$$

$$= c\int_0^{\infty} x^{\frac{n_1}{2}+k-1}\left[1+\left(\frac{n_1+n_2}{n_2}\right)x\right]^{-\,{(n_1+n_2)}/{2}} dx$$

$$= c\frac{n_2}{n_1+n_2}\int_0^{\infty}\left(\frac{n_2}{n_1+n_2}\,t\right)^{\frac{n_1}{2}+k-1}(1+t)^{-\,{(n_1+n_2)}/{2}}\,dt, \qquad \left[\text{taking } \left(\frac{n_1+n_2}{n_2}\right)x = t\right]$$

$$= c\left(\frac{n_2}{n_1+n_2}\right)^{\frac{n_1}{2}+k}\int_0^{\infty} t^{\left(\frac{n_1}{2}+k\right)-1}(1+t)^{-[(\frac{n_1}{2}+k)+(\frac{n_1}{2}-k)]}\,dt$$

$$= \frac{\left(\frac{n_1+n_2}{n_2}\right)^{\frac{n_1}{2}}\left(\frac{n_2}{n_1+n_2}\,t\right)^{\frac{n_1}{2}+k}}{B\left(\frac{n_1}{2},\frac{n_2}{2}\right)}\, \mathbf{x}\; B\left(\frac{n_1}{2}+k,\frac{n_2}{2}-k\right)\ \left[\text{putting the value of } c = \frac{\left({n_1+n_2}/{n_2}\right)^{\frac{n_1}{2}}}{B\left(\frac{n_1}{2},\frac{n_2}{2}\right)}\right]$$

$$= \left(\frac{n_2}{n_1+n_2}\right)^k\frac{\Gamma(\frac{n_1}{2}+\frac{n_2}{2})}{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})}\frac{\Gamma(\frac{n_1}{2}+k)\Gamma(\frac{n_2}{2}-k)}{\Gamma(\frac{n_1}{2}+k+\frac{n_2}{2}-k)}$$

$= \left(\frac{n_2}{n_1 + n_2}\right)^k \frac{\Gamma(\frac{n_1}{2}+k)\Gamma(\frac{n_2}{2}-k)}{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})}$, where, $n_2 > 2k$.

Note: The above integrals do not converge when $n_2 \le 2k$

4.5. Moment Generating Function:

A $\dot{R}$ random variable does not possess a moment-generating function. We have proven that the kth moment of $\dot{R}$ exists only when $n_2 > 2k$, but a random variable must possess a moment-generating function for all finite $k \in N$.

4.6. Relationship between *t* and the $\dot{R}$ - distribution:

In $\dot{R}$ distribution with (n_1, n_2) d.f., if we take $n_1 = 1$, $n_2 = \nu$, then the probability differential of $\dot{R}$ transforms to

$$dG(x) = \frac{\left(\frac{1+\nu}{\nu}\right)^{\frac{1}{2}}}{B\left(\frac{1}{2}, \frac{\nu}{2}\right)} . x^{\frac{1}{2}-1} \left[1 + \left(\frac{1+\nu}{\nu}\right) x\right]^{-{(1+\nu)}/{2}}$$

Again, taking $(1 + \nu)x = t^2$ i.e. $(1 + \nu)\,dx = 2t\,dt$, we obtain

$$dG(t) = \frac{\left(\frac{1+\nu}{\nu}\right)^{\frac{1}{2}}}{B\left(\frac{1}{2}, \frac{\nu}{2}\right)} \left(\frac{t^2}{1+\nu}\right)^{\frac{1}{2}-1} \left[1 + \frac{t^2}{\nu}\right]^{-{(1+\nu)}/{2}} \frac{2t}{1+\nu}\ dt \quad , \quad 0 \le t^2 < \infty .$$

$$= \frac{1}{B\left(\frac{1}{2}, \frac{\nu}{2}\right)\sqrt{\nu}} \frac{1}{\left[1+\frac{t^2}{\nu}\right]^{{(1+\nu)}/{2}}}\ dt , -\infty < t < \infty .$$

[The factor 2 disappears since the total integral in the range $(-\infty, \infty)$ is unity.]

This is the probability function of the Student's *t*-distribution with ν d.f. Hence we have the following relation between the *t* and $\dot{R}$ distributions.

If a statistic t follows Student's *t*-distribution with νd.f, then t^2 follows the $\dot{R}$ −distribution with (1, ν) d.f multiplied by $(1 + \nu)$ .Symbolically,

If $t \sim t_\nu$, then $t^2 \sim (1 + \nu)\dot{R}_{(1,\nu)}$.

With the help of the above relation, all the uses of the *t*- distribution can be regarded as applications of the $\dot{R}$ - distribution, i.e. instead of computing $t = \frac{\bar{x}-\mu}{S/\sqrt{n}}$, we may compute $\dot{R} = \frac{n(\bar{x}-\mu)^2}{(1+n)S^2}$ andthen apply $\dot{R}-$ test with (1, n) d.f. and so on.

4.7. Relationship between F and the $\dot{R}$ distribution:

$$\dot{R} \sim \frac{n_1}{n_1 + n_2} F(n_1, n_2)$$

These two distributions involve ratios of chi-square variables. Although they look similar, they represent different statistical distributions due to how the denominators are constructed. In **$\dot{R}$-** distribution, the numerator is still chi-square distributed, but it is scaled by a larger denominator than what is expected in F-distribution.

4.8. Special property (1): If $\dot{R} = \frac{n_1 n_2}{(n_1+n_2)^2}\frac{1}{\dot{R}'}$, then $\dot{R}'$ is a $\dot{R}(n_2, n_1)$ variate.

Proof: We have $\dot{R} = \frac{n_1 n_2}{(n_1+n_2)^2}\frac{1}{\dot{R}'}$, $d\dot{R} = -\frac{n_1 n_2}{(n_1+n_2)^2}\frac{d\dot{R}'}{\dot{R}'^2}$.

$$dP\left(\dot{R}'\right) = \frac{\left(\frac{n_1+n_2}{n_2}\right)^{\frac{n_1}{2}} \left[\frac{n_1 n_2}{(n_1+n_2)^2}\frac{1}{\dot{R}'}\right]^{\frac{n_1}{2}-1} \left| -\frac{n_1 n_2}{(n_1+n_2)^2}\frac{d\dot{R}'}{\dot{R}'^2} \right|}{B\left(\frac{n_1}{2},\frac{n_2}{2}\right)\left[1+\left(\frac{n_1+n_2}{n_2}\right).\frac{n_1 n_2}{(n_1+n_2)^2}\frac{1}{\dot{R}'}\right]^{(n_1+n_2)/2}}$$

$$= \frac{\left(\frac{n_1+n_2}{n_2}\right)^{\frac{n_1}{2}} \frac{n_1 n_2}{(n_1+n_2)^2}^{\frac{n_1}{2}} d\dot{R}'(\dot{R}')^{1-\frac{n_1}{2}}}{B\left(\frac{n_1}{2},\frac{n_2}{2}\right)\left[1+\frac{n_1}{n_1+n_2}\frac{1}{\dot{R}'}\right]^{(n_1+n_2)/2}} = \frac{\left(\frac{n_1+n_2}{n_2}\right)^{\frac{n_1}{2}} \left[\frac{n_1 n_2}{(n_1+n_2)^2}\right]^{\frac{n_1}{2}} d\dot{R}'(\dot{R}')^{1-\frac{n_1}{2}}}{B\left(\frac{n_1}{2},\frac{n_2}{2}\right)\left[\frac{n_1}{n_1+n_2}\frac{1}{\dot{R}'}\right]^{(n_1+n_2)/2}\left[1+\left(\frac{n_1+n_2}{n_1}\right)(\dot{R}')\right]^{(n_1+n_2)/2}}$$

$$= \frac{\left(\frac{n_1+n_2}{n_2}\right)^{\frac{n_1}{2}} \left[\frac{n_1 n_2}{(n_1+n_2)^2}\right]^{\frac{n_1}{2}} d\dot{R}'(\dot{R}')^{-1-\frac{n_1}{2}+\frac{n_1}{2}+\frac{n_2}{2}} \left[\frac{n_1}{n_1+n_2}\right]^{\frac{n_1}{2}-\frac{n_2}{2}}}{B\left(\frac{n_1}{2},\frac{n_2}{2}\right)\left[1+\left(\frac{n_1+n_2}{n_1}\right)(\dot{R}')\right]^{(n_1+n_2)/2}}$$

$$= \frac{\left(\frac{n_1+n_2}{n_1}\right)^{\frac{n_2}{2}} (\dot{R}')^{\frac{n_2}{2}-1} d\dot{R}'}{B\left(\frac{n_1}{2},\frac{n_2}{2}\right)\left[1+\left(\frac{n_1+n_2}{n_1}\right)(\dot{R}')\right]^{(n_1+n_2)/2}} \approx \dot{R}(n_2, n_1).$$

Therefore $\dot{R}' = \frac{n_1 n_2}{(n_1+n_2)^2 \dot{R}(n_1, n_2)}$ is a $\dot{R}(n_2, n_1)$ variate.

4.9. Special property (2): When $n_1 = 2$, the significance level of $\dot{R}$ corresponding to a significant probability p is $\dot{R} = \left(p^{-\frac{2}{n_2}} - 1\right)\frac{n_2}{2+n_2}$

Proof: $dP(\dot{R}) = \frac{\left(\frac{2+n_2}{n_2}\right)^{n_1/2}}{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})}\frac{\Gamma(\frac{2+n_2}{2})}{\left(1+\frac{n_1+n_2}{n_2}\dot{R}\right)^{\frac{n_1+n_2}{2}}}d\dot{R} = \frac{\Gamma\left(1+\frac{n_2}{2}\right)(1+\frac{2}{n_2})}{\Gamma(1)\Gamma\left(\frac{n_2}{2}\right)(\frac{2+n_2}{n_2})^{1+\frac{n_2}{2}}\left(\frac{n_2}{2+n_2}+\dot{R}\right)^{1+\frac{n_2}{2}}}d\dot{R} =$

$\frac{\frac{n_2}{2}(\frac{n_2}{2+n_2})^{\frac{n_2}{2}}}{\left(\frac{n_2}{2+n_2}+\dot{R}\right)^{1+\frac{n_2}{2}}}d\dot{R}.$

Hence, $p = \int_{\dot{R}}^{\infty} f\left(\dot{R}\right)d\dot{R} = \frac{n_2}{2}(\frac{n_2}{2+n_2})^{\frac{n_2}{2}}\int_{\dot{R}}^{\infty}\frac{d\dot{R}.}{\left(\frac{n_2}{2+n_2}+\dot{R}\right)^{1+\frac{n_2}{2}}}$

So, $p = [\frac{\frac{n_2}{2+n_2}}{\left(\frac{n_2}{2+n_2}+\dot{R}\right)}]^{\frac{n_2}{2}}.$

$\Rightarrow p^{-\frac{2}{n_2}} = 1 + \left(1+\frac{2}{n_2}\right)\dot{R}.$

$\Rightarrow \dot{R}. = \left(p^{-\frac{2}{n_2}} - 1\right)\frac{n_2}{2+n_2}.$

4.10. Special property (3):

As $n_2 \to \infty$, $\dot{R} \to 0$ .i.e. $\lim_{n_2\to\infty} \dot{R} = \lim_{n_2\to\infty}\frac{\frac{\chi_1^2}{n_1+n_2}}{\frac{\chi_2^2}{n_2}} = 0$ (In probability)

Proof: As $n_2 \to \infty$, $\frac{\chi_2^2}{n_2} \to 1$ (By the law of Large Numbers)

$\Rightarrow \chi_2^2 \approx n_2.$

Again, as $n_2 \to \infty$, $\frac{n_2}{n_1+n_2} \to 1$.

So, $\dot{R} = \frac{\frac{\chi_1^2}{n_1+n_2}}{\frac{\chi_2^2}{n_2}} = \left(\frac{n_2}{n_1+n_2}\right)\frac{\chi_1^2}{\chi_2^2} \approx 1.\frac{\chi_1^2}{n_2} \to 0$ (In probability)

5. Assumptions for S.B.s' $\dot{R}$- test: (for the equality of population variances)

The following assumptions are made in the S.B.s' $\dot{R}$- test:

i) The parent population from which the sample is drawn is normal.

ii) The sample observations are independent.

iii) The population standard deviation σ is unknown.

iv) In preparing the test, the variation that is bigger between two variables, shouldn't be considered.

6. $\dot{R}$- distribution applications:

6.1. $\dot{R}$- test for equality of population variances**.**

Suppose that we want to test whether two independent samples $x_i, (i = 1, 2, \dots \dots \nu_1)$ and $y_j, (j = 1, 2, \dots \dots \nu_2)$ have been drawn from normal populations with the same variances σ^2 (say).

Under the null hypothesis (*H₀*) **:** $\sigma_x^2 = \sigma_y^2 = \sigma$, i.e. the population variances are equal, the statistic $\dot{R}$ is given by $\dot{R} = \frac{\dot{K}.\, S_X^2}{S_Y^2}$**.**

Where $\dot{K}$ is a constant $= \frac{\nu_1 - 1}{\nu_1 + \nu_2 - 2}$ [this is known to be a $\dot{R}$ constant] andwhere

$S_X^2 = \frac{1}{\nu_1 - 1} \sum_{i=1}^{\nu_1} (x_i - \bar{x})^2$, and $S_Y^2 = \frac{1}{\nu_2 - 1} \sum_{j=1}^{\nu_2} (y_j - \bar{y})^2$ are unbiased estimates of population variance σ^2 obtained from two independent samples and follow $\dot{R}$ distribution with $(\nu_1 - 1, \nu_2 - 1)$ d.f.

Proof (1): $\dot{R} = \frac{\dot{K}.\, S_X^2}{S_Y^2} = \frac{\dot{K}\left[\frac{\nu_1}{\nu_1 - 1} s_x^2\right]}{\left[\frac{\nu_2}{\nu_2 - 1} . s_y^2\right]} = \frac{\left[\frac{\nu_1 s_x^2}{\sigma_x^2} \cdot \frac{\dot{K}}{\nu_1 - 1}\right]}{\left[\frac{\nu_2 s_y^2}{\sigma_y^2} \cdot \frac{1}{\nu_2 - 1}\right]} = \frac{\left[\left(\frac{\nu_1 s_x^2}{\sigma_x^2}\right) \cdot \frac{1}{(\nu_1 - 1) + (\nu_2 - 1)}\right]}{\left[\frac{\nu_2 s_y^2}{\sigma_y^2} \cdot \frac{1}{\nu_2 - 1}\right]}$ by setting the value of $\dot{K}$.

Since $\frac{\nu_1 s_x^2}{\sigma_x^2}$ and $\frac{\nu_2 s_y^2}{\sigma_y^2}$ are independent chi-square variables with $\nu_1 - 1$ $and\ \nu_2 - 1$ d.f. respectively, $\dot{R}$ follows **S.B.s'** **$\dot{R}$-** distribution with $(\nu_1 - 1, \nu_2 - 1)$ d.f.

Proof (2): We may prove this with the help of the maximum likelihood function.

Let us take two normal populations $N\ (\mu_1\ ,\ \sigma_1^2\ )$ and $N\ (\mu_2\ ,\ \sigma_2^2\ )$ where the means μ_1, μ_2 and variances $\sigma_1^2,\ \sigma_2^2$ are unknown. We want to test this hypothesis

(*H₀*): $\sigma_1^2 = \sigma_2^2 = \sigma$ unknown with $\mu_1\ and\ \mu_2$ unspecified against the alternative hypothesis (*H₁*): $\sigma_1^2 \neq \sigma_2^2$ with $\mu_1\ and\ \mu_2$ unspecified.

If x_{1i}, (i = 1, 2,…., m) and $x_{2\,j}$, (j = 1, 2, …., n) are two independent random samples of sizes m and n from $N\ (\mu_1,\ \sigma_1^2\ )$ and $N\ (\mu_2,\ \sigma_2^2\ )$ respectively then, the likelihood function is

$$L = (\frac{1}{2\pi\,\sigma_1^2})^{m/2} \exp[-\frac{1}{2\,\sigma_1^2}\sum_{i=1}^{m}(x_{1i}-\mu_1)^2] \text{ x } (\frac{1}{2\pi\,\sigma_2^2})^{n/2} \exp[-\frac{1}{2\,\sigma_2^2}\sum_{j=1}^{n}(x_{2\,j}-\mu_2)^2] \text{---}(\mathbf{I})$$

In this case, the parameter space $\Omega = \{\ (\mu_1,\ \mu_2\ ,\ \sigma_1^2\ ,\ \sigma_2^2);\ -\infty < \mu_i < \infty\ ;\ \ \sigma_i^2 > 0, (i = 1, 2\ )\}$

The subspace $\omega = \{(\mu_1,\ \mu_2\ ,\ \sigma^2):-\infty < \mu_i < \ \infty\ ;\ \ (i = 1, 2\ ),\ \ \sigma^2\ > 0\ \}$

The maximumlikelihood estimates for $\mu_1,\ \mu_2\ ,\ \sigma_1^2\ ,\ \sigma_2^2$ are given by the equations

$$\left.\begin{aligned}
&\frac{\partial}{\partial\mu_1}\log L\ = 0\ \Rightarrow\ \ \hat{\mu}_1 = \frac{1}{m}\sum_{i=1}^{m} x_{1i} =\ \bar{x}_1\ .\\
&\frac{\partial}{\partial\mu_2}\log L\ = 0\ \Rightarrow\ \hat{\mu}_2\ = \frac{1}{n}\sum_{j=1}^{n} x_{2\,j} =\ \bar{x}_2\ .\\
&\frac{\partial}{\partial\sigma_1^2}\log L\ = 0\ \Rightarrow\ \ \hat{\sigma}_1^2 = \frac{1}{m}\sum_{i=1}^{m}(x_{1i}-\bar{x}_1)^{\ 2} =\ s_1^2\ \ (say)\ .\\
&\frac{\partial}{\partial\sigma_2^2}\log L\ = 0\ \Rightarrow\ \ \hat{\sigma}_2^2 = \frac{1}{n}\sum_{j=1}^{n}(x_{2\,j}-\ \bar{x}_2)^{\ 2} =\ s_2^2\ \ (say)\ .
\end{aligned}\right\}\text{--------------}(\mathbf{II})$$

Putting in equation --- (**I**), we obtainthe maximum of L in the parameter space Ω as --

$$L(\hat{\Omega}) = \left(\frac{1}{2\pi s_1^2}\right)^{m/2} (\frac{1}{2\pi s_2^2})^{n/2} e^{-(m+n)/2} \text{ ------------------------------ } (\mathbf{III})$$

In the parameter subspaceω of Ω, the likelihood function is given by

$$L(\omega) = \left[\frac{1}{2\pi\ \sigma^2}\right]^{(m+n)/2}.\exp[-\frac{1}{2\ \sigma^2}\ \{\sum_i(x_{1i}-\mu_1)^2\ +\sum_j\left(x_{2\,j}-\mu_2\right)^2\ \}\ ] \text{ ---- } (\mathbf{IV})$$

The MLEs for $\mu_1\ ,\mu_2\ \ and\ \ \sigma^2$ are given by

$$\hat{\mu}_1 =\ \bar{x}_1\ ,\ \hat{\mu}_2 =\ \bar{x}_2 \text{ --------------------------- } (\mathbf{V})$$

and $\sigma^2 = \frac{1}{(m+n)}\ [\ \sum_i(x_{1i}-\ \hat{\mu}_1)^2\ +\ \ \sum_j\left(x_{2\,j}-\hat{\mu}_2\right)^2\ \ ]$

$$= \frac{1}{(m+n)}\ [\ \sum_i(x_{1i}-\ \bar{x}_1)^2\ +\ \ \sum_j\left(x_{2\,j}-\ \bar{x}_2\right)^2\ \ ] = \frac{ms_1^2+ns_2^2}{m+n} \text{ ------- } (\mathbf{VI})$$

Substituting from (**V**) and (**VI**) in (**IV**), we obtain the maximum of L in the parameter subspace ω as -------

$$L(\hat{\omega}) = [\frac{m+n}{2\pi(m\, s_1^2 + n\, s_2^2)}]^{(m+n)/2} \cdot e^{-(m+n)/2}$$

The criterion for the likelihood ratio test (λ) is given by ------

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})} = (m+n)^{(m+n)/2} \{\frac{(s_1^2)^{m/2}(s_2^2)^{n/2}}{(m\, s_1^2 + n\, s_2^2)^{(m+n)/2}}\}$$

$$= \frac{(m+n)^{(m+n)/2}}{m^{m/2} n^{n/2}} \{\frac{(ms_1^2)^{m/2}(ns_2^2)^{n/2}}{(m\, s_1^2 + n\, s_2^2)^{(m+n)/2}}\} \text{--------------- } \textbf{(VII)}$$

We know that under H_0, the statistic $\dot{\boldsymbol{R}} = \frac{\frac{\Sigma (x_{1i} - \bar{x}_1)^2 (m-1)}{(m-1)(m+n-2)}}{\Sigma (x_{2j} - \bar{x}_2)^2 / (n-1)} = \frac{S_1^2}{S_2^2} \dot{K}$------------ (**VIII**)

$$= \frac{\frac{m}{m-1} s_1^2}{\frac{n}{n-1} s_2^2} \dot{K}, \quad where \ \frac{m}{m-1} s_1^2 = S_1^2 \ and \ \frac{n}{n-1} s_2^2 = S_2^2$$

$$= \frac{m(n-1)s_1^2}{n(m-1)s_2^2} \dot{K}.$$

Therefore, $\frac{m\, s_1^2}{ns_2^2} = \frac{(m-1)}{K(n-1)} \dot{\boldsymbol{R}} = \frac{(m+n-2)}{(n-1)} \dot{\boldsymbol{R}}$

Substituting in (**VII**) and simplifying, we obtain $\lambda = \frac{(m+n)^{(m+n)/2}}{m^{m/2} n^{n/2}} \{\frac{[\frac{(m+n-2)}{(n-1)} \dot{R}]^{m/2}}{[1 + \frac{(m+n-2)}{(n-1)} \dot{R}]^{(m+n)/2}}$

This follows that λ is a monotonic function of $\dot{\boldsymbol{R}}$ hence, the test can be carried out with $\dot{\boldsymbol{R}}$**,** defined in (**VIII**) as the test statistic.

6.2. The <u>critical values of the $\dot{\boldsymbol{R}}$-Distributions are as follows</u>:

The region of rejection of H_0 when H_0 is true is that region of the outcome set where H_0 is rejected if the sample point falls in that region and is called the critical region.The size of the critical region is α, the probability of committing a type I error i.e., rejecting a true null hypothesis.

The available $\dot{\boldsymbol{R}}$-Table (given in Table: 01 and Table: 02 in this article) gives the critical values of the $\dot{\boldsymbol{R}}$ right-tailed test, i.e., the critical region is determined by the right areas. Thus, the significant values $\dot{\boldsymbol{R}}_\alpha(n_1, n_2)$ at the level of significance α and (n_1, n_2) **d.f** are determined by -----

$P[\dot{\boldsymbol{R}} > \dot{\boldsymbol{R}}_{\alpha}(\mathbf{n_1}, \boldsymbol{n_2})] = \alpha$, as shown in Figure: 02.

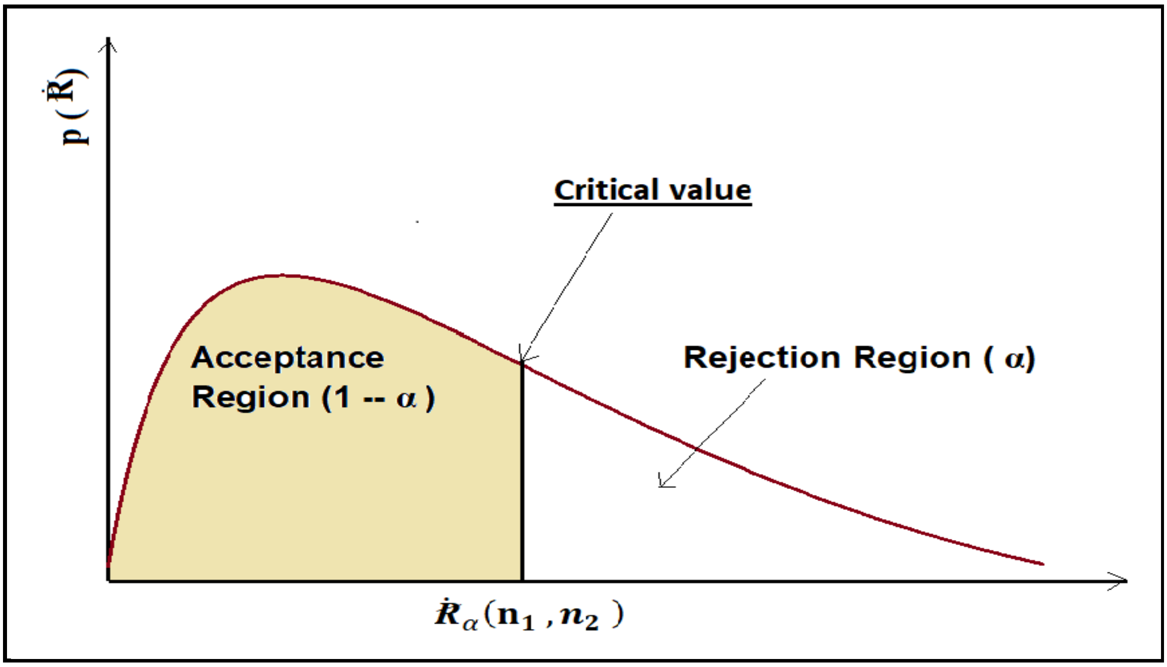

Figure: 02: Critical values of the $\dot{R}$- distribution.

6.3. Type I Error (α): Rejection of a true null hypothesis.

$$\dot{\boldsymbol{R}}(\mathbf{n_1}, \boldsymbol{n_2}) \sim \mathbf{c}.\,\mathbf{F}((\mathbf{n_1}, \boldsymbol{n_2})\ , \boldsymbol{where}\ \boldsymbol{c} = \frac{n_1}{\mathrm{n}_1 + \mathrm{n}_2}.$$

$$\mathbf{P}\ (\dot{\boldsymbol{R}} < c.\,\boldsymbol{F}\alpha_{/2}(\mathbf{n_1}, \boldsymbol{n_2})\mathbf{or}\ \dot{\boldsymbol{R}} > \ c.\,\boldsymbol{F}_{1\,-}\alpha_{/2}(\mathbf{n_1}, \boldsymbol{n_2})\ \ |\boldsymbol{H_0}\ ) = \boldsymbol{\alpha}$$

So, the Type I Error rate is still $\boldsymbol{\alpha}$.

Probability of Type I error = Level of significance = Size of critical region.

Type II Error ($\boldsymbol{\beta}$): Acceptance of a false null hypothesis. (Type II Error rate)

Suppose the true variance ratio is $r = \frac{\sigma_1^2}{\sigma_2^2} \neq 1$.Then under this, the $\dot{\boldsymbol{R}}(\mathrm{n}_1, n_2)$distribution becomes a non-central F-distribution and hence

$$\dot{\boldsymbol{R}}(\mathrm{n}_1, n_2) \sim \mathbf{c.r.F}((\mathrm{n}_1, n_2)\mathbf{.}$$

Therefore, $\beta = \mathbf{P}(\boldsymbol{c}.\boldsymbol{F}\alpha_{/2}(\mathrm{n}_1, n_2) \leq \dot{\mathbf{R}} \leq \boldsymbol{c}.\boldsymbol{F}_{1\,-}\alpha_{/2}(\mathrm{n}_1, n_2)\ \ |\boldsymbol{H_1})$.

This can be calculated numerically for given values of $\mathrm{n}_1, n_2, \boldsymbol{r}, \boldsymbol{\alpha}$.

Thepowerofa statistical testis theprobability that it correctly rejects the null hypothesis $H_0$when thealternative hypothesis H_1 is true**.**

Power = 1- Probability of Type II error = 1 – β

= Probability of rejecting H_0 when H_1 is true.

Example: 01. In one sample of 8 observations the sum of the squares of the deviations of the sample values from the sample mean was 84.4 and in another sample of 10 observations it was 102.6. Test where this difference is significantat the 5% level.

Solution: Here, $\nu_1 = 8, \nu_2 = 10, \sum(x - \bar{x})^2 = 84.4, \sum(y - \bar{y})^2 = 102.6$,

$\dot{\boldsymbol{R}}$ constant $\dot{K} = 0.4375$, $\mathbf{S_X^2} = \frac{\mathbf{1}}{\nu_1 - \mathbf{1}}\sum_{i=1}^{\nu_1}(x_i - \bar{x})^2 = \frac{84.4}{7} = 12.057$ and

$$\mathbf{S_Y^2} = \frac{\mathbf{1}}{\nu_2 - \mathbf{1}}\sum_{j=1}^{\nu_2}\left(y_j - \bar{y}\right)^2 = \frac{102.6}{9} = 11.4$$

Under $H_0 = \sigma_x^2 = \sigma_y^2 = \sigma$, i.e.the estimates of σ^2given by the sampler are equal, the test statistic is

$$\dot{\boldsymbol{R}} = \frac{\dot{K}.\ \mathbf{S_X^2}}{\mathbf{S_Y^2}} = \frac{(0.4375)(12.057)}{11.4} = 0.4624.$$

Tabulated $\dot{\boldsymbol{R}}_{0.05}$ (7, 9) d.f is 0.0018. [From Table: 02, given below in this paper.]

Since $\dot{\boldsymbol{R}} > \dot{\boldsymbol{R}}_{0.05}$ H_0 may be rejected at the 5% significance level.

Example: 02. Two random samples yielded the following values:

Sample	size	sum of squares of deviations fromthe mean
1	5	80
2	6	200

Test whether the samples come from two normal populations with equal variances.

Solution: The null hypothesis is as follows: Two samples are drawn from two normal populations with the same variances, i.e. $H_0 = \sigma_1^2 = \sigma_2^2 = \sigma$.

Here we may apply the $\dot{\boldsymbol{R}}$**-**test to testthe equality of variances.

We are given$\nu_1 = 5$, $\nu_2 = 6$, $\sum(x - \bar{x})^2 = 80, \sum(y - \bar{y})^2 = 200$

$\mathbf{S_X^2} = \frac{\mathbf{1}}{\nu_1 - \mathbf{1}}\sum_{i=1}^{\nu_1}(x_i - \bar{x})^2 = \frac{80}{4} = 20$, $\mathbf{S_Y^2} = \frac{\mathbf{1}}{\nu_2 - \mathbf{1}}\sum_{j=1}^{\nu_2}\left(y_j - \bar{y}\right)^2 = \frac{200}{5} = 40.$

Here Ќ= 4/9 = 0.44444. Therefore, $\dot{\boldsymbol{R}} = \frac{\text{Ќ. } \mathbf{S_X^2}}{\mathbf{S_Y^2}} = \frac{(0.4444)(20)}{40} = 0.2222$

The tabulated value of $\dot{\boldsymbol{R}}_{0.05}$ (4,5) d.f. is 9.24. [From Table: 02, given below in this paper.]

Since $\dot{\boldsymbol{R}} < \dot{\boldsymbol{R}}_{0.05}$ H_0 may be accepted at a 5% level of significance.

Again, Tabulated value of $\dot{\boldsymbol{R}}_{0.01}$ (4,5) d.f. is 16.11. [From Table: 03, given below in this paper.]

Since $\dot{\boldsymbol{R}} < \dot{\boldsymbol{R}}_{0.01}$ H_0 may be accepted at the 1% significance level.

Example: 03 Two random samples are drawn from two populations and the following observations were obtained ------

Sample I: (x) 15 17 18 19 20 21 22 24 26 28

Sample II: (y) 15 19 22 23 24 25 26 28 30 32 34 36 37

Find the variances of two samples and test whether the two populations have the same variance.

Solution: Here, $\bar{x}$ = 21 and $\bar{y}$ = 27, $\nu_1 = 10$, $\nu_2 = 13$, $\sum(x - \bar{x})^2 = 156$, $\sum(y - \bar{y})^2 = 524$

$\mathbf{S_X^2} = \frac{1}{\nu_1 - 1}\sum_{i=1}^{\nu_1}(x_i - c)^2 = \frac{156}{9} = 17.33$, $\mathbf{S_Y^2} = \frac{1}{\nu_2 - 1}\sum_{j=1}^{\nu_2}\left(y_j - \bar{y}\right)^2 = \frac{524}{12} = 43.66.$

Ќ = 0.4285. $\dot{\boldsymbol{R}} = \frac{\text{Ќ. } \mathbf{S_X^2}}{\mathbf{S_Y^2}} = \frac{(0.4285)(17.33)}{43.66} = 0.17012.$

The tabulated value of $\dot{\boldsymbol{R}}_{0.05}$ (9, 12) d.f. is 0.024. [From Table: 02, given below in this paper.]

Since $\dot{\boldsymbol{R}} > \dot{\boldsymbol{R}}_{0.05}$ H_0 may be rejected at the5% level of significance.

Therefore, we conclude that the two populations may not have the same variance.

Alternative procedure:

If we interchange the values of x and y experimentally, then we have

$\bar{x}$ = 27 and $\bar{y}$ = 21, $\nu_1 = 13$, $\nu_2 = 10$, $\sum(x - \bar{x})^2 = 524$, $\sum(y - \bar{y})^2 = 156$

$\mathbf{S_X^2} = \frac{1}{\nu_1 - 1}\sum_{i=1}^{\nu_1}\left(x_i - \bar{x}\right)^2 = 43.66$, $\mathbf{S_Y^2} = \frac{1}{\nu_2 - 1}\sum_{j=1}^{\nu_2}\left(y_j - \bar{y}\right)^2 = 17.33.$

Ќ = 0.57, ⇒ $\dot{\boldsymbol{R}} = \frac{\text{Ќ. } \mathbf{S_X^2}}{\mathbf{S_Y^2}} = \frac{(0.57)(43.66)}{17.33} = 1.43$

Tabulated value of $\dot{\boldsymbol{R}}_{0.05}$ (12, 9) d.f is 0.00011.

Since $\dot{\boldsymbol{R}} > \dot{\boldsymbol{R}}_{0.05}$ H_0 may be rejected at the5% significancelevel.

Therefore, we conclude that the two populations may not have the same variance.

Remarks: In all three above examples, the greater variance has beentaken in the denominator but this is not acceptable in the F-distribution.

7. Comparisons between $\dot{R}$ and F distributions: (for testing the equality of population variances).

i) In the F-distribution, generally the greater of the two variances is to be taken in the numerator and n_1 corresponds to the greater variance by convention, to ensure the F-ratio ≥ 1 under H_1, making it easier to compare with right-tailed critical values and this increases sensitivity. large F-values strongly indicate inequality of variances. However, this is not the case in the $\dot{R}$- test.In this test, everyone can choose which variance will be in the numerator and which variance is in the denominator.

ii) In particular, when the sample size is very small, the $\dot{R}$- test is more appropriate than the F- test.

iii) The mode of $\dot{R}$ - test is always smaller than that of the F-test. This difference grows larger as n_2 increases. For example, let $n_1 = 10$, $n_2 = 20$. Then

Mode (F) = 0.727, Mode ($\dot{R}$) ≃ 0.242, a significant leftward shift in the distribution.

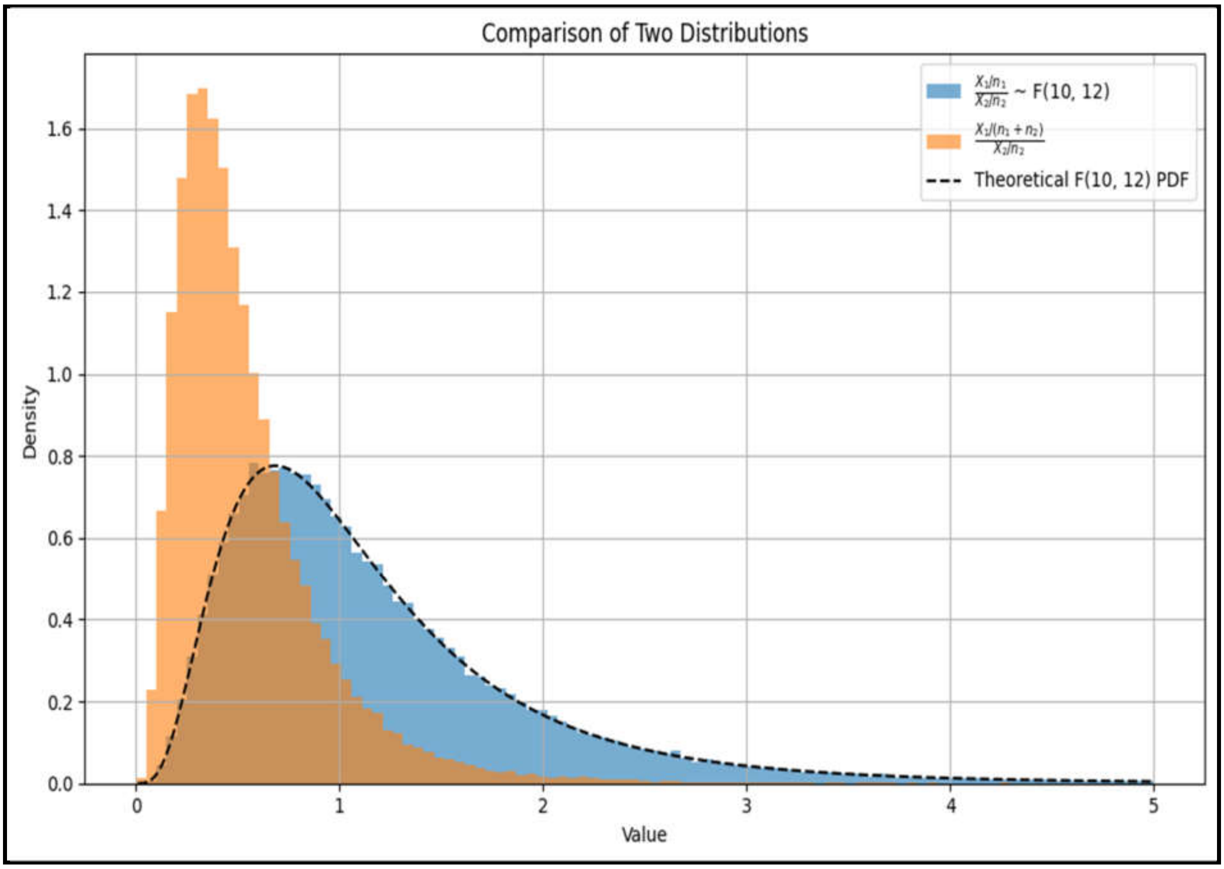

Figure: 03. Comparative histograms of two distributions with n_1=10, n_2= 12 degreesof freedom.

.

iv) From the figure.03, we have the following observations ------ -

- The blue histogram represents the distribution of $\frac{\frac{\chi_1{}^2}{n_1}}{\frac{\chi_2{}^2}{n_2}}$. This aligns clearly with the F-distribution curve (black dashed line). It is expected since it is an F(10, 12) distribution.
- The orange histogram represents the distribution of $\frac{\frac{\chi_1{}^2}{n_1+n_2}}{\frac{\chi_2{}^2}{n_2}}$. This distribution is compressed towards zero. It lies to the left of the standard F-distribution because the numerator is being divided by a larger numerator than in the F-distribution.
- The peak of the orange distribution is higher and shifted leftward (maximum values are close to zero).
- This shift to the left in the distribution of $\dot{R}$ makes it less likely to detect large values.
- The spread is narrower with fewer high outliers compared to the F-distribution
- The standard F-distribution has a larger right tail, indicating more extreme values due to less compression in the numerator.
- The $\dot{R}$ curve is leptokurtic.

8. Results of simulations: [Among F-, $\dot{R}$-, Bartlett's and Levene's tests for (10, 20)d.f.] [Illustrated in table 01 and figure 04]

Table 01: Final observations (10,000 Simulations, Normal data)

Test Statistics	mean	Mode	Variance	Sensitivity	Robustness	Assumes Normality
F	1.425	1.32	0.297	High	Low	Yes
$\dot{R}$	0.693	0.61	0.158	Medium	Low	Yes
Bartlett's	0.989	0.75	1.949	Very High	Very Low	Yes
Levene's	0.971	0.83	0.581	Low	High	Data may be non-normal

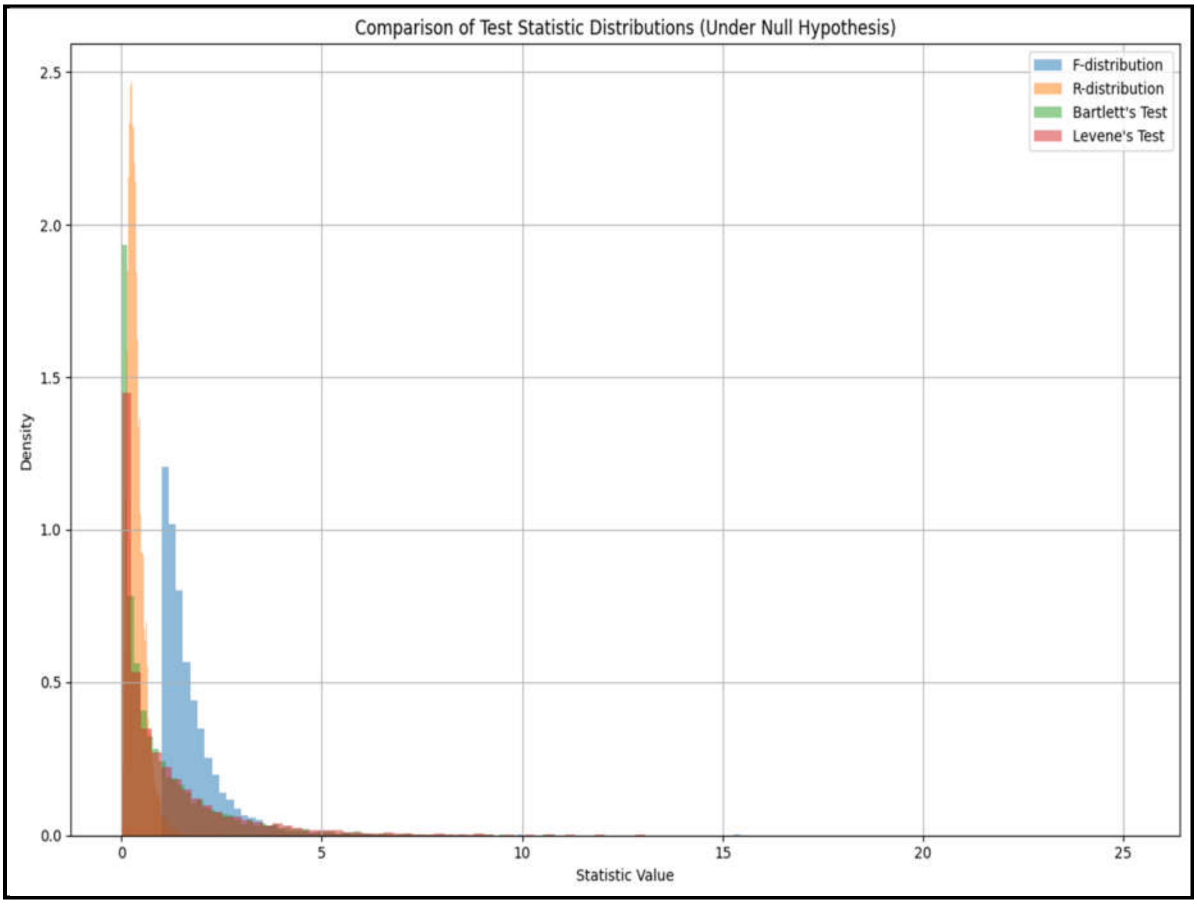

Figure 04: Simulations Histograms of four distributions for (10, 20) degrees of freedom.

9. Summary and concluding remarks:

In this work, a new sampling distribution SBs'$\dot{R}-$ distributionis proposedin Section 3 to overcome the limitation of the F- distribution. Various properties have been thoroughly discussed. The $\dot{R}$ - distribution is clearly positively skewed, andthe probability$p(\dot{R})$increases steadily at first until it reaches its peak (corresponding to the modal value which is less than 1) and then decreases slowly to become tangential at $\dot{R} = \infty$,i.e., the$\dot{R}$- axis isasymptotic to the right tail.

It is also clear from the above discussion that $\dot{R}-$ distribution is more accurate when the sample size is small i.e. in a real sense it is an exact sampling distribution.

The newly proposed $\dot{R}$- testmight be preferable in the following cases -----

- In building a model where one wants to suppress the influence of ${\chi_1}^2$ relative to ${\chi_2}^2$.
  In this case, reduced scale couldbe useful.
- In machine learning or empirical modelling, where we are not calculating formal influence but just balancing the magnitude.
- $\dot{R}$has afixed upper bound.$\dot{R} \leq \frac{\frac{{\chi_1}^2}{n_1}}{\frac{{\chi_2}^2}{n_2}} = F$andthe scaling factor $\frac{n_2}{\mathrm{n}_1 + \mathrm{n}_2}$keepsall values in a compressed range.
  This could be usefulin data compression, tax transformation, or feature scaling interpretability and boundedness like other statistical properties.
- In non-parametric settings or engineering control systems, we might create a ratio like $\dot{R}$to weigh evidence differently, e.g. putting less emphasis on volatile data from a small sample (numerator).
- This could be useful for any threshold decision.

I am leaving these theories for further research (for junior researchers / my students).

10. Appendix.

Table: 02 $\dot{R}$- distribution,
(Significant Points) Values of $\dot{R}_{0.05,\ n_1,n_2}$

n_1 \ n_2	1	2	3.	4	5	6	7	8	9	10	12
1	20.05	10.54	6.75	5.23	2.71	2.65	2.00	1.90	1.50	1.10	0.60
2	133	9.50	4.42	2.31	2.14	1.285	1.05	0.975	0.77	0.623	0.579
3	139.2	101.4	71.8	58.8	45.8	27.3	25.5	16.33	14.54	11.68	7.3
4	7.01	32.49	22.5	13.75	9.24	6.348	3.44	2.8	2.63	2.357	1.25
5	0.951	0.986	0.915	0.835	0.688	0.649	0.621	0.573	0.523	0.311	0.215
6	0.425	0.696	0.794	0.707	0.658	0.600	0.450	0.413	0,334	0.295	0.183
7	0.013	0.137	0.187	1.27	1.17	0.65	0.127	0.0022	0.0018	0.0015	0.0011
8	0.0025	0.0070	0.0725	0.163	0.151	0.138	0.100	0.051	0.009	0.0057	0.00058

9	0.091	0.11	0.145	0.189	0.238	0.19	0.165	0.145	0.1257	0.093	0.024
10	0.0018	0.007	0.012	0.0148	0.0167	0.018	0.0128	0.00785	0.00659	0.00130	0.000084
12	0.00006	0.000075	0.00076	0.0012	0.0015	0.00186	0.0019	0.0004	0.00011	0.0000824	0.000069

Table: 03 $\dot{R}$- distribution,

(Significant Points) Values of $\dot{R}_{\mathbf{0.01},\ \mathbf{n_1},\ \boldsymbol{n_2}}$

n_1 \ n_2	1	2	3	4	5	6	7	8	9	10	12
1	87.03	83.94	15.16	12.64	6.15	5.14	3.56	3.2	2.5	1.8	1.3
2	3333	49.5	12.33	6.00	2.96	2.73	2.12	1.712	1.458	1.283	0.994
3	169.1	155.4	118.4	103.5	72.5	43.5	36.2	29.5	22.6	18.52	17.43
4	21.8	66.167	66	22.77	16.11	11.76	6.8	3.53	2.98	2.88	2.70
5	1.003	1.267	.973	0.863	0.861	0.772	0.626	0.609	0..567	0.424	0.272
6	0.665	0.805	0.93	0.725	0.682	0.621	0,529	0.427	0.388	0.331	0.238
7	0.02	0.343	0.47	1.34	1.229	0.70	0.202	0.0089	0.0019	0.00155	0.00137
8	0.0053	0.013	0.1515	0.208	0.141	0.145	0.125	0.078	0.010	0.0069	0.0013
9	0.163	0.245	0.30	0.357	0.366	0.358	0.348	0.281	0.144	0.106	0.045
10	0.002	0.00014	0.165	0.0192	0.0204	0.0207	0.016	0.0097	0.0068	0.0036	0.00085
12	0.00013	0.0006	0.00128	0.0018	0.00189	0.00259	0.0026	0.0016	0.000236	0.000085	0.00007

- **Author contributions:** This research paper has only one author, so it is only his contribution.
- **Funding:** This research paper has received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.
- **Conflict of interest:** The author has declared that no competing interests exist.
- **Data availability**: I do not analyze or regenerate datasets from others, because my work proceeds via a theoretical and mathematical approach. One can obtain the relevant materials from the reference below.
- **Ethics, Consent to Participate, and Consent to Publish declarations**: Not applicable

Reference:

1. Bhattacharyya, S: Statistical Quality Control: Acceptance Plans in the Light of Fuzzy Mathematics. Journal of Statistical Theory and Applications, 170-212(2023), vol. 22(3),

https://doi.org/10.1007/s44199-023-00059-3

2. Fisher R.A. (1924). On a distribution yielding the error functions of several well-known statistics Proceedings of the International Mathematical Congress, Toronto, 805—813

3. Fisher R.A. (1925). Statistical methods for Research workers: London; Oliver and Boyd.

4. Fisher R.A. (1955). "Statistical Methods and Scientific Induction" Journal of the Royal Statistical Society, series B 17: 69 – 78.

5. Gentleman, W. M. and Jenkins, M. A. (1968), " An approximation to Student's*t* distributions". Biometrika, 55, 571—572

6. George, E. O. and Singh, K. P. (1987). An Approximation of F distribution by Binomial probabilities, Statistics and Probability Letters, 5, 169—173

7. Gray, H. L., Thompson, R. W, and Mc-Williams, G. V. (1969). A new approximation for the chi-square integral, Mathematics of Computation, 23, 85—89

8. Lancester, H.O.(1966); Forerunners of the Pearson chi-square, Aust. J. of statistics. 8, 117—126

9.Lancester, H.O.(1969). The Chi-squared Distribution, Wiley

10. Laurent, B; Massart, P; (2000), "Adaptive estimation of a quadratic functional by model solution". The Annals statistics 28(5) Doi:10.1214/aos/1015957395

11. Neyman, J; Pearson, E. S. (1983). “On the Problem of the Most Efficient Tests of Statistical Hypotheses” Philosophical Transactions of the Royal Society A. 231 (694-706); (289-337),

doi:10.1098/rsta.1933.0009.

12. Plaekett, R. L (1983). Karl Pearson and the Chi-squared test, Int. Stat. Rev. 51(1): 59-72.

13. Shoraek, G. R.(2000). Probability for statisticians, Springer – Verlag, New York