A Study on Recent New Results on Some Graph Valued Functions

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Abstract.

In this paper, the result on some valued function(digraph operator), namely the block line cut vertex digraph BLC(D) of a digraph D is defined, and the problem of reconstructing a digraph from its block line cut vertex digraph is presented. Outer planarity, maximal outer planarity, and minimally non-outer planarity properties of these digraphs are discussed.

Keywords: Planar And Nonplanar Graphs, Cutvertex, Line Graph, Wheel Graph, Total Blict Graph

INTRODUCTION

All graphs considered here are finite, undirected and without loops or multiple edges. The edges, cut vertices and blocks of a graph G are called its members. Two blocks B_i and B_j are adjacent if they have common cutvertex.

Definition 1.1 The Edge degree of an edge uv in G is the number of the edges adjacent to edge uv or deg u + deg v − 2. A Block vertex is a vertex in TBn (G) corresponding to a block of G.

Definition 1.2 A graph is said to be Planar if it can be embedded in a plane so that no two edges intersect. Otherwise, the graph is nonplanar.

A maximal planar graph is one to which no edge can be added without losing planarity. The concept of outerplanar graphs was studied by Tang [27]. A planar graph is said to be outerplanar if it can be embedded in a plane so that all its vertices lie on the same region. Otherwise the graph is nonouterplanar. An outerplanar graph G is maximal outerplanar if no edge can be added without losing outerplanarity. Chartrand and Harary [2] obtained a characterization of outerplanar graphs in terms of forbidden subgraphs.

Definition 1.3 The concept of non-zero inner vertex number of a planar graph was introduced by Kulli [11]. A nonnegative integer r such that any plane embedding of a planar graph G has at least r vertices not lying on the boundary of the exterior region of G is called the inner vertex number of G, denoted as i(G) and this indicates that G has r inner vertices. In general, the planar graphs having i(G) = r, r > 0, are called r-nonouterplanar graphs. In particular, zero nonouterplanar graphs are outerplanar graphs. 1-nonouterplanar graphs will be called minimally nonouter planar graphs. For these graphs i(G) = 1. This concept has been extensively studied by Kulli [11] and others.

Definition 1.4 The Line graph of a graph G, denoted L (G), is the graph whose vertices are the edges of G, with two vertices of L(G) adjacent whenever the corresponding edges of G are adjacent. The concept of the Line graph of a given graph is so natural that it has been independently discovered by many authors giving different name.

Definition 1.5 The crossing number C(G) of a graph G is the minimum number of pair wise intersections (or crossings) of its edges when G is drawn in the plane. Obviously, C(G) = 0 if and only if G is planar. If C(G) =1, then G is said to have crossing number one.

Definition 1.6 A vertex v of G is called a cut vertex if its removal produces a disconnected graph. That is, G-v has at least two components.

Definition 1.7 A Wheel graph W_n is a graph with n vertices formed by connecting a single vertex to all vertices of an (n-1) cycle.

All undefined terms may be referred to Harary [8].

We need the following theorems for the proof of our further results.

Theorem 1.1[8]: If G is a graph (V,E) whose vertices have degree d_i, then Line graph L(G) has E vertices and E_L edges, where $E_L = -E + \frac{1}{2}\sum d_i^2$

Theorem 1.2[25]: The line graph L(G) of a graph G is planar if and only if G is planar, the degree of each vertex of G is atmost 4 and every vertex of degree 4 is a cutvertex.

Theorem 1.3[4]: The Line graph L(G) of graph G is outerplanar if and only if the degree of each vertex of G is atmost 3 and every vertex of degree 3 is a cutvertex.

Theorem 1.4[12]: The Line graph of G has crossing number one if and only if G is planar and (i) or (ii) holds.

(i) The maximum degree $\Delta(G)$ is 4 and there is a unique non-cutvertex of degree 4.

(ii) The maximum degree $\Delta(G)$ is 5, every vertex of degree 4 is a cut vertex, there is a unique vertex of degree 5 and it has atmost 3 edges in any block.

Theorem 1.5[8]: A graph G(V,E) is planar if and only if $|E| \le 3|V| - 6$.

Theorem 1.6[8]: If G is a nontrivial connected graph with $|V|$ vertices which is not a path, then $L^n(G)$ is Hamiltonian for all $n \ge |V| - 3$.

MAIN RESULTS

Definition 2.1 Total Blict graph TBn (G) of a graph G is the graph whose vertex set is the union of the set of edges, set of cut vertices and set of blocks of G in which two vertices are adjacent if and only if the corresponding members of G are adjacent or incident except the adjacency of cut vertices. In Figure 1.1, a graph G and its Total Blict graph TBn (G) are shown.

Remark 2.1: For any graph G, L (G) $\subset$ TBn (G).

Remark 2.2: For any cycle C_v, $V \ge 3$, i [TBn (G)] ≥ 1.

In particular i [TBn (C_3)] = 1.

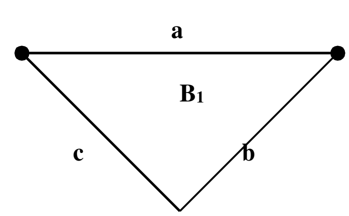
Remark 2.3: For every non-separable graph G, TBn (G) is a block.

Remark 2.4: Every bridge in G forms a pendant edge in TBn (G).

Remark 2.5: For any non-separable graph G an edge ‘a’ with edge degree odd corresponds to the vertex ‘a’ in TBn (G) whose vertex degree is even and vice versa.

Remark 2.6: For any separable graph G an edge ‘a’ incident to the cutvertex corresponds to the vertex ‘a’ of odd degree in TBn (G).

Remark 2.7: For any graph G, TBn (G) is a bridgeless graph.



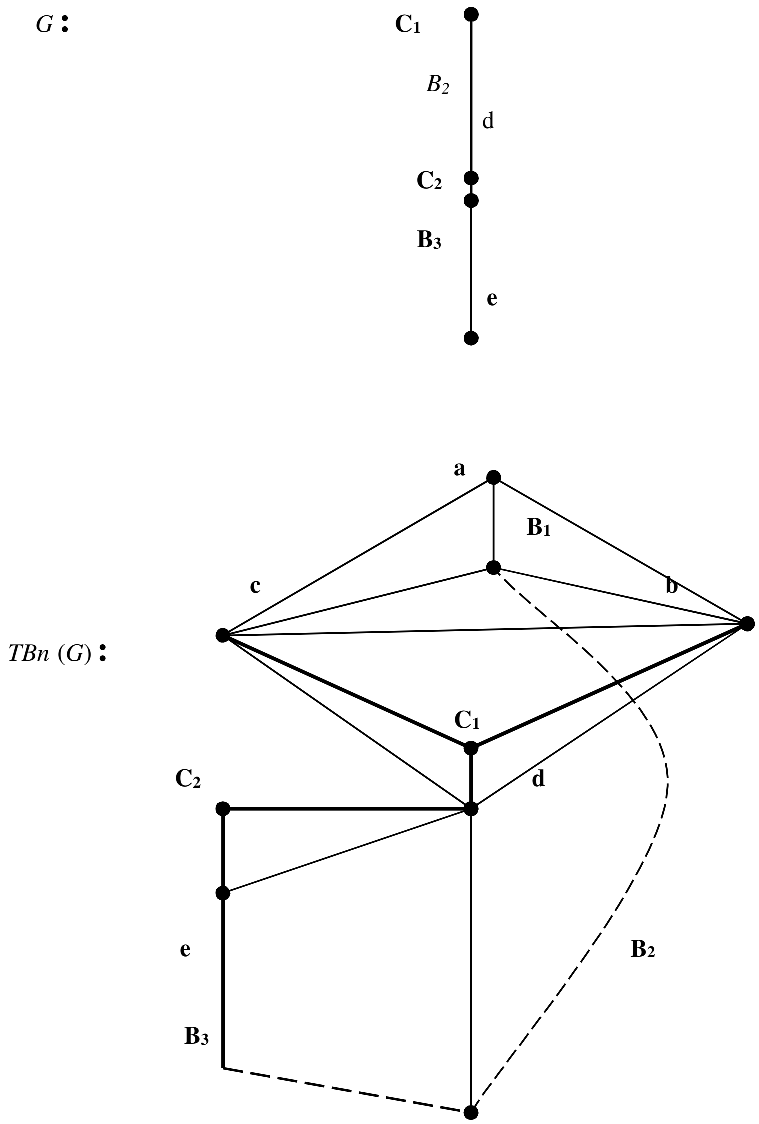


Figure 1.1

Theorem 2.1: For any nontrivial connected (V, E) graph G whose vertices have degree d_i, C is the number of the cutvertices in G, B_k be the number of blocks then TBn (G) has (E + B_k + C) vertices and $\frac{1}{2}\sum_{i=1}^{v} d_i^2 + \sum_{j=1}^{c} \deg C_j + \sum_{i,j=1,i\neq j}^{k} B_{i,j}$ edges, where C_j is the jth cutvertex. $B_{i,j}$ denotes that B_i is adjacent to B_j .

Proof: By the definition of TBn (G), the number of vertices is ($E + B_k + C$). For the number of edges, since $L(G) \subset TBn(G)$, by Theorem 1.1[8], $-E + \frac{1}{2}\sum d_i^2$ edges are contributed to TBn (n). By definition, every block vertex is adjacent to vertices corresponding to edges from which it is formed in G. This gives E edges to TBn (G). Every cutvertex is adjacent to the vertices corresponding to the edges incident to it in G. This adds $\sum_{j=1}^{c} \deg C_j$ edges to TBn (G). These blocks B_i adjacent to B_j for $i \neq j$ gives $\sum_{i,j=1,i\neq j}^{k} B_{i,j}$ edges this adds to the total number of edges to TBn (G).

Hence the number of edges in TBn (G) is given by

$$E[TBn(G)] = -E + \frac{1}{2}\sum_{i=1}^{v} d_i^2 + E + \sum_{j=1}^{c} \deg C_j + \sum_{i,j=1,i\neq j}^{k} B_{i,j}$$

$$E[TBn(G)] = \frac{1}{2}\sum_{i=1}^{v} d_i^2 + \sum_{j=1}^{c} \deg C_j + \sum_{i,j=1,i\neq j}^{k} B_{i,j}$$

Hence the proof.

In the following theorem we establish the planarity of TBn (G).

Theorem 2.2: The Total Blict graph TBn (G) of graph G is planar if and only if $\Delta(G) \leq 3$ and every vertex of degree 3 is a cut vertex.

Proof: Suppose TBn (G) is planar. Assume $\Delta(G) > 3$. Let v be a vertex of degree 4 in G, we have the following cases.

Case 1: If v is a non cutvertex, then the number of edges incident to v forms $< K_4 >$ as a subgraph in L (G). By definition of TBn (G) the block vertex is adjacent to all the vertices of $< K_4 >$ which gives $< K_4 > \subset TBn\ (G)$ which is non planar, a contradiction.

Case 2: If v is a cutvertex then the number of edges incident to v forms $< K_4 >$ as a subgraph in L (G). By the definition, the cutvertex V is adjacent to each vertex of $< K_4 >$ gives $< K_4 > \subset TBn\ (G)$, a contradiction for planarity of TBn (G).

Suppose $\Delta(G) \le 3$ and G has a non-cutvertex v of degree 3. Clearly v lies on exactly one block. Then by Theorem 1.3[4], $i[L(G)] \ge 1$. Let B_k be the block vertex belonging to the block where v lies in TBn (G). B_k is adjacent to all the vertex of L (G), which gives at least one crossing and the adjacencies of blocks gives more crossings. Hence TBn (G) is nonplanar, a contradiction.

Conversely, suppose $\Delta(G) \le 3$ and every vertex of degree 3 is a cutvertex. We have the following cases.

Case 1: If every cutvertex of the degree 3 lies on 3 blocks of G, then clearly G is a tree. Each block of $TBn(G) \cap [B_k]$ is either K_3 *or* K_4. Since each block of G is an edge, each vertex of $TBn(G) \cap [B_k + \sum C_j]$ is incident with an end edge gives each block of TBn (G) either K_2 *or* K_3 *or* K_4. The adjacent of blocks in TBn (G) gives K_3 as sub graphs. Hence TBn (G) is planar.

Case 2: If every cut vertex of degree 3 lies on 2 blocks, then every block of G is either K_2 or C_v, $v \ge 3$. In TBn (G) $(E + B_k + C_j)3 - 6 \ge \frac{1}{2}\sum_{i=1}^{v} d_i^2 + \sum_{j=1}^{c} \deg C_j + \sum_{i,j=1, i \ne j}^{k} B_{i,j}$ by theorem 1.5[8], TBn (G) is planar. Hence the theorem is proved.

In the next theorem we obtain a condition for the Total Blict graph to be outer planar.

Theorem 2.3: The Total Blict graph TBn (G) of a graph G is outerplanar if and only if G is a path.

Proof: Suppose TBn (G) is outerplanar. Assume G has a vertex v of degree 3. We consider the following cases.

Case 1: If v is a cutvertex and lies on 3 blocks, then $K_{1,3}$ is a subgraph of G and $L[K_{1,3}] = K_3$. In TBn (G) vertex v is adjacent to all the vertices of K_3 forming $< K_4 >$ as a subgraph which is nonouter planar, a contradiction. Hence G has no vertex of degree 3.

Case 2: Suppose G is a cycle C_v where $v \ge 3$, for $v = 3$ by definition, TBn (G) contains K_4 which is nonouter planar. From case 1 and case 2, G should be a path.

Conversely, If G is a path, then TBn (G) is outerplanar.

If G is P_1, then TBn(G) is also P_1. If G is P_2 then TBn (G) is Cr_5 which is outerplanar. For every addition of an edge to P_2, we get an addition of $Cr_5 - X$ to TBn (G) of P_2, where X is the common edge in TBn (G), which is outer planar. In general for every addition of an edge to the path gives (n - 1) times $Cr_5 - X$ in TBn (G). Hence it is outer planar.

Theorem 2.4: For any graph G with P > 2 vertices, the Total Blict graph TBn (G) is not maximal outerplanar.

Proof: We prove the theorem by two cases,

Case 1: If G consists a cycle C_3, Then TBn (G) = K_4 which is non-outerplanar. There is nothing to discuss further.

If G consists a cycle C_p, p > 3, the vertex of TBn (G) corresponding to the block in G spoils the outer planarity of TBn (G).

Case 2: If G consists a path P_2, then TBn (G) consists C_4 which is a non-maximal outer planar.

Theorem 2.5: The Total Blict graph TBn (G) of a graph G is minimally nonouter planar if and only if G is a cycle C_3. **Proof:** Suppose Total Blict graph TBn (G) is minimally nonouter planar. Assume $\Delta(G) \geq 3$. We consider the following cases.

Case 1: If $\Delta(G) > 3$, then by theorem 2.2 TBn (G) is nonplanar, a contradiction.

Case 2: If $\Delta(G) = 3$ we have the following subcases.

Subcase (i): If G is a tree and has more than one vertex of degree 3. Then G contains more then one $K_{1,3}$ as subgraph. Each $K_{1,3}$ in G gives K_3 in L(G) and adjacency of blocks and edges incident to cutvertices gives a graph which contains K_4 in TBn(G). Hence $i[TBn\ (G)] > 1$, a contradiction.

Subcase (ii): If G is not a tree and has more than one vertex of degree 3. Then each vertex of degree 3 in G gives a subgraph $<K_4>$ in TBn(G). Adjacency of blocks with the edges and itself gives $i[TBn(G)] > 1$, a contradiction. ***Subcase (iii):*** If G is not a tree, has cutvertex v of degree 3 which lies on 2 blocks of G. Clearly one block is K_2 and other is $C_v\,(v \geq 3)$. Let e_1, e_2, e_3 be the edges incident on v and $e_1 \in \langle K_2 \rangle$, $e_2, e_3 \in C_v\,(v \geq 3)$. In TBn (G) e_1, e_2, e_3 together with v form $\langle K_4 \rangle$, along with adjacency of blocks gives non outer planar graph with $i[TBn(G)] > 1$, a contradiction.

Conversely, Suppose G is a cycle C_3 then by Remark 2.2, $i[TBn(G)] = 1$. Hence TBn (G) is minimally nonouterplanar.

Theorem 2.6: The Total Blict graph TBn (G) of a graph G has crossing number one if and only if G is planar and $\Delta(G) \leq 3$ for every vertex v of G, G has exactly two adjacent non cutvertices of degree 3.

Proof: Suppose G is planar and $\Delta(G) \leq 3$ for every vertex v of G, G has exactly two adjacent non cut vertices of degree 3. Then G has a block homeomorphic to $K_4 - X$ or G has a block $K_4 - X$ as a sub graph. In each case, let e = uv be an edge incident on two adjacent non cut vertices of degree 3 which lies in the interior region of $K_4 - X$.

In L(G), L($K_4 - X$) gives a block homeomorphic to W_5 or a block W_5 as a sub graph in L(G). In TBn (G) the adjacency of inner vertex of W_5 and vertex corresponding to block gives one crossing. Hence $c[TBn(G)] = 1$

Conversely, if TBn (G) has crossing number one then G is planar.

Case 1: Suppose G has crossing number one. Let $\Delta(G) > 3$. By Theorem 2.2, TBn (G) is nonplanar, a contradiction. Therefore $\Delta(G) \leq 3$.

Case 2: If G has atleast two cutvertices of degree 3.

Let v_1 and v_2 be the two cut vertices of degree 3 incident on e_1, e_2, e_3 and f_1, f_2, f_3 edges respectively. Hence L(G) has two induced sub graphs as C_3. Since v_1 is incident on e_1, e_2, e_3 and v_2, is incident on f_1, f_2, f_3 by definition of TBn (G), v_1 is adjacent to each vertex of C_3 which gives $K_4 - X$ as a sub graph in TBn (G). Adjacency of block vertex with $K_4 - X$ gives $K_5 - X$ as a sub graph in TBn (G). Hence TBn (G) has two induced sub graphs as $K_5 - X$. Clearly, $c[TBn(G)] > 1$, a contradiction. Hence G has only one cut vertex of degree 3.

Suppose TBn (G) has crossing number one and exactly one cut vertex of degree 3. Then TBn (G) contains $< K_4 >$ as a sub graph and G must contain a $K_{1,3}$ as a sub graph which is planar.

Hence the proof.

Theorem 2.7: For any graph G, TBn (G) is non Eulerian.

Proof: Proof of this theorem is obvious by Remark 2.5 and Remark 2.6.

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