Bounds on Eccentric Version of ISI index of graphs

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Abstract

In this paper, we introduce the inverse sum indeg eccentric index ξ_{ISI} of a graph G, so that it is the sum of the terms $\frac{\epsilon_x \epsilon_y}{\epsilon_x+\epsilon_y}$ for the edges xy in G, where ϵ_x is the eccentricity of the vertex x in graph G. Relationaships between $\xi_{ISI}(G)$ and other topological indices are derived using well-known inequalities.

Keywords: topological index, eccentricity index, inverse sum indeg index, bounds.

1 Introduction

Eccentricity in graph theory measures how far a vertex is from the most distant vertex in the graph. It helps in understanding network structures and has applications in areas like transportation, communication, and biological networks.

Eccentricity-based topological indices are numerical values derived from a graph's structure that remain unchanged regardless of labeling or representation. These indices, such as the Wiener index and Zagreb eccentricity index, play a significant role in chemical graph theory. They help predict chemical compound properties based on molecular structure. The Wiener index, introduced by H. Wiener in 1947, was one of the first indices used to study the properties of chemical compounds.

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The Zagreb eccentricity index, inspired by Zagreb indices but using eccentricity instead of degree, is given by:

$$\xi_z(G) = \sum_{x \in V(G)} \epsilon_x^2$$

Here, degree is a local property, while eccentricity provides a global perspective on a graph's structure.

Sharma, Goswami, and Madan [10] introduced the eccentric connectivity index, combining adjacency and distance measures. This index has been widely used in mathematical models predicting biological activity. Over the years, researchers have refined and expanded these indices to consider additional factors, such as the position of heteroatoms in molecules [5].

In the past two decades, numerous studies have explored the mathematical properties and applications of eccentricity-based indices. These indices provide insights into graph structure and have practical applications in chemistry, network analysis, and data science. For further details, see [7].

Table 1: Eccentricity-based Topological Indices

Topological Index	**Notations**	**Mathematical Expression**
First Zagreb eccentric index[12]	$\xi_{M_1}(G)$	$\sum_{xy \in E(G)} [\epsilon_x + \epsilon_y]$
Second Zagreb eccentric index[12]	$\xi_{M_2}(G)$	$\sum_{xy \in E(G)} [\epsilon_x \epsilon_y]$
Eccentric connective index	$\xi^{ce}(G)$	$\sum_{x \in V(x)} \frac{d(x)}{\epsilon(x)}$
First general Zagreb index[8]	$\xi^{\alpha}_{M_1}(G)$	$\sum_{xy \in E(G)} (\epsilon_x^{\alpha-1} + \epsilon_y^{\alpha-1})$
Inverse sum indeg eccentric index	$\xi_{ISI}(G)$	$\sum_{xy \in E(G)} \frac{\epsilon_x \epsilon_y}{\epsilon_x + \epsilon_y}$
Generalization of Zagreb index	$\xi_{\alpha,\beta}(G)$	$\sum_{xy \in E(G)} \frac{(\epsilon_x \epsilon_y)^{\alpha}}{(\epsilon_x + \epsilon_y)^{\beta}}$
Harmonic Index [11]	$\xi_H(G)$	$\sum_{xy \in E(G)} \frac{2}{\epsilon_x + \epsilon_y}$
General sum connectivity eccentric index	$\xi^{\alpha}_{\chi}(G)$	$\sum_{xy \in E(G)} (\epsilon_x + \epsilon_y)^{\alpha}$
Sigma eccentric index	$\xi_{\sigma}(G)$	$\sum_{xy \in E(G)} (\epsilon_x - \epsilon_y)^2$
Albertson eccentric index[1]	$\xi_{Al}(G)$	$\sum_{xy \in E(G)} \lvert\epsilon_x - \epsilon_y\rvert$
General Randic eccentric index [2]	$\xi^{\alpha}_{R}(G)$	$\sum_{xy \in E(G)} (\epsilon_x \epsilon_y)^{\alpha}$
General F-eccentric index[6]	$\xi^{\alpha}_{F}(G)$	$\sum_{xy \in E(G)} (\epsilon_x^2 + \epsilon_y^2)^{\alpha}$

2 Bounds for ξ_{ISI}

In this section, we obtain the upper and lower bounds for inverse sum indeg eccentric index of a graph G.

Theorem 2.1. *Let G be (n,m) graph with $n \geq 2$ $\xi_{ISI}(G) \leq \frac{m^2}{2\rho(G)\xi_R^{-1}(G)}$. The bound is sharp and the self-centered graph satisfies it.*

Proof. Let G be (n,m) graph with $n \geq 2$. By Cauchy-schwartz inequality we obatin,

$$\begin{aligned}
\left(\sum_{xy\in E(G)} \sqrt{\frac{\epsilon_x\epsilon_y}{\epsilon_x+\epsilon_y}}\sqrt{\frac{\epsilon_x+\epsilon_y}{\epsilon_x\epsilon_y}}\right)^2 &\leq \sum_{xy\in E(G)}\left(\sqrt{\frac{\epsilon_x+\epsilon_y}{\epsilon_x\epsilon_y}}\right)^2 \sum_{xy\in E(G)}\left(\sqrt{\frac{\epsilon_x\epsilon_y}{\epsilon_x+\epsilon_y}}\right)^2\\
m^2 &= \sum_{xy\in E(G)}\left(\frac{\epsilon_x+\epsilon_y}{\epsilon_x\epsilon_y}\right)\sum_{xy\in E(G)}\left(\frac{\epsilon_x\epsilon_y}{\epsilon_x+\epsilon_y}\right)\\
m^2 &\leq 2\rho(G)\sum_{xy\in E(G)}(\epsilon_x\epsilon_y)^{-1}\xi_{ISI}(G)\\
m^2 &\leq 2\rho(G)\xi_R^{-1}(G)\xi_{ISI}(G).
\end{aligned}$$

This implies that

$$\xi_{ISI}(G) \geq \frac{m^2}{2\rho(G)\xi_R^{-1}(G)}.$$

Hence

$$\xi_{ISI}(G) \geq \frac{m^2}{2\rho(G)\xi_R^{-1}(G)}. \tag{2.1}$$

To show that the inequality of Equation (2.1) is sharp, let G be a self-centered graph with $\rho(G) = r(G)$. Then

$$\begin{aligned}
\sum_{\epsilon_x\epsilon_y\in E(G)}\frac{\epsilon_x\epsilon_y}{\epsilon_x+\epsilon_y}\sum_{\epsilon_x\epsilon_y\in E(G)}\frac{\epsilon_x+\epsilon_y}{\epsilon_x\epsilon_y} &= \sum_{\epsilon_x\epsilon_y\in E(G)}\frac{(r(G))^2}{2r(G)}\sum_{\epsilon_x\epsilon_y\in E(G)}\frac{2r(G)}{(r(G))^2}\\
&= \left(\frac{mr(G)}{2}\right)\left(\frac{2m}{r(G)}\right)\\
&= m^2
\end{aligned}$$

This implies that,

$$2\rho(G)\xi_R^{-1}(G)\xi_{ISI}(G) = 2r(G)\left(\frac{m^2}{(r(G))^2}\right)\left(\frac{(r(G))^2}{2r(G)}\right) = m^2.$$

On the other hand, let

$$2\rho(G)\xi_R^{-1}(G)\xi_{ISI}(G) = m^2.$$

Also we have

$$\frac{m^2(r(G))^3}{\rho^3(G)} \leq 2\rho(G)\xi_R^{-1}(G)\xi_{ISI}(G) \leq \frac{m^2\rho^3(G)}{(r(G))^3}.$$

with equality if and only if $r(G) = \rho(G)$.

■

The Proof of the converse of Cauchy-Schwarz inequality proved by Matinez-Perez et.al.[9] which is given below.

Lemma 2.2. *If* $\alpha_i, \beta_j \geq 0$ *and* $\theta\beta_j \leq \alpha_j \leq \Omega\beta_j$ *for* $1 \leq j \leq k$*, then*

$$\left(\sum_{j=1}^{k}\alpha_j^2\right)^{1/2}\left(\sum_{j=1}^{k}\beta_j^2\right)^{1/2} \leq \frac{1}{2}\left(\sqrt{\frac{\Omega}{\theta}}+\sqrt{\frac{\theta}{\Omega}}\right)\sum_{j=1}^{k}\alpha_j\beta_j. \tag{2.2}$$

If $\alpha_j > 0$ *for some* $1 \leq j \leq k$*, then the equality holds* $\iff$ $\theta = \Omega$ *and* $\alpha_j = \theta b_j$ *for every* $1 \leq j \leq k$.

Theorem 2.3. *If* G *is a graph with* m *edges, then* $\xi_{ISI}(G) \geq \frac{\rho(G)^{3/2}r(G)^{3/2}}{(\rho^3(G)+r^3(G))m}\xi_{M_2}(G)\xi_H(G)$ *with equality if and only if* G *is self-centered graph.*

Proof. By Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\xi_{M_2}(G) = \sum_{xy\in E(G)}\epsilon_x\epsilon_y &\leq \left(\sum_{xy\in E(G)}(\epsilon_x\epsilon_y)^2\right)^{1/2}\left(\sum_{xy\in E(G)}(1)\right)^{1/2}\\ \xi_{M_2}(G)m^{-1/2} &\leq \left(\sum_{xy\in E(G)}(\epsilon_x\epsilon_y)^2\right)^{1/2}.\end{aligned}$$

$$\frac{\xi_H(G)}{2} = \sum_{xy\in E(G)}\frac{1}{\epsilon_x+\epsilon_y} \leq \left(\sum_{xy\in E(G)}\left(\frac{1}{\epsilon_x+\epsilon_y}\right)^2\right)^{1/2}\left(\sum_{xy\in E(G)}(1)\right)^{1/2}.$$

Hence

$$\left(\sum_{xy\in E(G)}\left(\frac{1}{\epsilon_x+\epsilon_y}\right)^2\right)^{1/2} \geq \frac{m^{-1/2}\xi_H(G)}{2}.$$

For every $xy \in E(G)$,

$$2r^3(G) \leq \epsilon_x\epsilon_y(\epsilon_x+\epsilon_y) = \frac{\epsilon_x\epsilon_y}{\frac{1}{\epsilon_x+\epsilon_y}} \leq 2\rho^3(G).$$

This Lemma (2.2) gives that,

$$\frac{1}{2}\left(\sqrt{\frac{\rho^3(G)}{r^3(G)}}+\sqrt{\frac{r^3(G)}{\rho^3(G)}}\right)\sum_{xy\in E(G)}\frac{\epsilon_x\epsilon_y}{\epsilon_x+\epsilon_y} \geq \left(\sum_{xy\in E(G)}(\epsilon_x\epsilon_y)^2\right)^{1/2}\left(\sum_{xy\in E(G)}\left(\frac{1}{\epsilon_x+\epsilon_y}\right)^2\right)^{1/2}.$$

$$\sum_{xy\in E(G)}\frac{\epsilon_x\epsilon_y}{\epsilon_x+\epsilon_y} \geq \frac{\left(\sum_{xy\in E(G)}(\epsilon_x\epsilon_y)^2\right)^{1/2}\left(\sum_{xy\in E(G)}\left(\frac{1}{\epsilon_x+\epsilon_y}\right)^2\right)^{1/2}}{\frac{1}{2}\left(\sqrt{\frac{\rho^3(G)}{r^3(G)}}+\sqrt{\frac{r^3(G)}{\rho^3(G)}}\right)}.$$

Thus,

$$\begin{aligned}\xi_{ISI}(G) &\geq 2\left(\sqrt{\frac{r^3(G)}{\rho^3(G)}}+\sqrt{\frac{\rho^3(G)}{r^3(G)}}\right)\xi_{M_2}(G)\frac{\xi_H(G)}{2}\\ &= \frac{\rho(G)^{3/2}r(G)^{3/2}}{(\rho^3(G)+r^3(G))m}\xi_{M_2}(G)\xi_H(G). \qquad (2.3)\end{aligned}$$

If G is self centered graph, then

$$\frac{r(G)^{3/2+3/2}}{(2r^3(G))m}\xi_{M_2}(G)\xi_H(G)=\left(\frac{r^3(G)}{2r^3(G)m}\right)(m\rho^2(G))\left(\frac{m}{\rho(G)}\right)=\frac{m\rho(G)}{2}=\xi_{ISI}(G).$$

If the equality (2.3) is attained, then by Lemma 2.2 we have, $2r^3(G)=2\rho^3(G)$ and G is self-centered graph. ■

Lemma 2.4. *For every real number $p>0, x_k\geq 0, a_k>0$ for every $1\leq k\leq n$,*

$$\sum_{k=1}^{n}\frac{x_k^{p+1}}{a_k^p} \geq \frac{\left(\sum\limits_{i=1}^{n}x_k\right)^{p+1}}{\left(\sum\limits_{k=1}^{n}a_k\right)^{p}}.$$

Equality holds if and only if $\frac{x_1}{a_1}=\frac{x_2}{a_2}=\ldots=\frac{x_n}{a_n}$.

Theorem 2.5. *For any graph G, $\xi_{ISI}(G)\geq\frac{r^2(G)m^2}{\xi_{M_1}(G)}$ with equality if and only if G is self centered with $\epsilon_x+\epsilon_y$ is constant for all $xy\in E(G)$.*

Proof. By the definition of ξ_{ISI}, we have

$$\begin{aligned}\xi_{ISI}(G) &= \sum_{xy\in E(G)}\frac{\epsilon_x\epsilon_y}{\epsilon_x+\epsilon_y}\\ &\geq r^2(G)\sum_{xy\in E(G)}\frac{1}{\epsilon_x+\epsilon_y}\\ &\geq r^2(G)\frac{\left(\sum\limits_{xy\in E(G)}\sqrt{1}\right)^2}{\sum\limits_{xy\in E(G)}(\epsilon_x\epsilon_y)}\\ &= \frac{r^2(G)m^2}{\xi_{M_1}(G)} \qquad (2.4)\end{aligned}$$

Suppose the equality holds in Equation (2.4). In this case by Lemma 2.4, G is a self centred and $\epsilon_x+\epsilon_y$ is constant for every $xy\in E(G)$. Conversely, if $\epsilon_x+\epsilon_y$ is constant for all $xy\in E(G)$ for a self centered graph, we can easily see that equality in (2.4). ■

Theorem 2.6. *For a graph G, $\xi_{ISI}(G)\leq\frac{\xi_{M_1}(G)}{4}$ with equality holds if and only if G is self-centered graph.*

Proof. From arithmetic-harmonic mean inequality, we obtain

$$\begin{aligned}\xi_{ISI}(G) &= \sum_{xy\in E(G)} \frac{\epsilon_x\epsilon_y}{\epsilon_x+\epsilon_y}\\ &\leq \frac{1}{4}\sum_{xy\in E(G)}\left(\frac{\epsilon_x\epsilon_y}{\epsilon_x}+\frac{\epsilon_x\epsilon_y}{\epsilon_y}\right)\\ &= \frac{1}{4}\sum_{xy\in E(G)}(\epsilon_y+\epsilon_x)\\ &= \frac{1}{4}\xi_{M_1}(G). \qquad (2.5)\end{aligned}$$

Suppose that equality holds in (2.5). Then for any $xy \in E(G)$, $\epsilon_x = \epsilon_y$. Hence one can check that the equality holds in (2.5) if and only if G is self centered graph. Conversely, let G be a self centered graph. Then $\epsilon_x = \epsilon_y = r(G), \forall xy \in E(G)$. Thus, $\xi_{ISI}(G) = \sum\limits_{xy\in E(G)} \frac{\epsilon_x\epsilon_y}{\epsilon_x+\epsilon_y} = \frac{mr(G)}{2}$. and $\frac{\xi_{M_1}(G)}{4} = \frac{2r(G)m}{4} = \frac{r(G)m}{2}$.

This gives the theorem. ■

Theorem 2.7. *For a graph G, $\xi_{ISI}(G) \geq \frac{r^3(G)m^2}{\xi_2(G)+r^2(G)m}$. Equality holds if and only if G is a self centered and $\epsilon_x+\epsilon_y$ is constant for all $xy \in E(G)$.*

Proof. Since $\epsilon_x, \epsilon_y \geq r(G)$, we have $(\epsilon_x - r(G))(\epsilon_y - r(G)) \geq 0$.

This implies that

$$\begin{aligned}\epsilon_x\epsilon_y - r(G)(\epsilon_x+\epsilon_y) + r^2(G) &\geq 0\\ \implies \frac{\epsilon_x\epsilon_y + r^2(G)}{r(G)} &\geq \epsilon_x+\epsilon_y \qquad (2.6)\end{aligned}$$

Equality holds in (2.6) $\epsilon_x = r(G)$ (or) $\epsilon_y = r(G)$ (or) $\epsilon_x = \epsilon_y = r(G),\ \forall xy \in E(G)$. Hence

$$\begin{aligned}\xi_{ISI}(G) &\geq \sum_{xy\in E(G)} \frac{r(G)\epsilon_x\epsilon_y}{\epsilon_x\epsilon_y + r^2(G)}\\ &\geq \frac{r^2(G)\left(\sum\limits_{xy\in E(G)}\sqrt{1}\right)^2}{\left(\sum\limits_{xy\in E(G)}(\epsilon_x\epsilon_y + r^2(G)\right)}\\ &= \frac{r^3(G)m^2}{\xi_2(G)+r^2(G)m}. \qquad (2.7)\end{aligned}$$

Suppose equality holds in (2.7) Then all the inequalities in the above statement must be equalities. ■

By Lemma 2.2 we have $\epsilon_x+\epsilon_y$ is constant $\forall xy \in E(G)$ and G is self centered graph. Conversely, for a self centered graph G with $\epsilon_x+\epsilon_y$ is constant, $\forall xy \in E(G)$, it is easy to see that equality (2.5) holds.

Lemma 2.8. *(Schwetzers Inequality) Let* $a_1, a_2, ...a_n$ *be postive real numbers such that* $1 \le i \le n$ *holds* $m \le a_i \le M$. *Then* $\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} \frac{1}{a_i}\right) \le \frac{n^2(n+M)^2}{4nM}$ *with equality if and only if* $a_1 = a_2 = ... = a_{\frac{n}{2}} = m$ *and* $x_{\frac{n}{2}+1} = ... = x_n = M$.

Theorem 2.9. *For a graph* G, $\xi_{ISI}(G) \le \frac{m^2(r^2(G)+\rho^3(G))}{4\xi^{ce}(G)r^2(G)\rho^2(G)}$ *with equality if and only if* G *is self centered graph.*

Proof. Since, $\frac{r^2(G)}{\rho(G)} \le \frac{\epsilon_x\epsilon_y}{\epsilon_x+\epsilon_y} \le \frac{\rho^2(G)}{r(G)}, \quad \forall xy \in E(G)$. Using Schwetzers inequality, we have

$$\sum_{xy\in E(G)} \frac{\epsilon_x\epsilon_y}{\epsilon_x + \epsilon_y} \sum_{xy\in E(G)} \frac{\epsilon_x + \epsilon_y}{\epsilon_x\epsilon_y} \quad \le \quad \frac{m^2\left(\frac{r^2(G)}{\rho(G)} + \frac{\rho^2(G)}{r(G)}\right)}{4\left(\frac{r^2(G)}{\rho(G)}\right)\left(\frac{\rho^2(G)}{r(G)}\right)}.$$

This implies that

$$\xi_{ISI}(G)\left(\sum_{x\in V(G)} \frac{d(x)}{\epsilon(x)}\right) \le \frac{m^2\left(\frac{r^3(G)+\rho^3(G)}{\rho(G)r(G)}\right)}{4\left(\frac{r^2(G)\rho^2(G)}{\rho(G)r(G)}\right)}.$$

It gives

$$\xi_{ISI}(G)\xi^{ce}(G) \le \frac{m^2}{4r^2(G)\rho^2(G)}(r^3(G) + \rho^3(G)).$$

Hence

$$\xi_{ISI}(G) \le \frac{m^2(r^3(G) + \rho^3(G))}{4\xi^{ce}(G)(r^2(G)\rho^2(G))}.$$

Equality holds if and only if G is self centered. ■

Theorem 2.10. *For a real number* α *and a graph* G,

$$\xi_{ISI}(G) \le \begin{cases} \frac{F^\epsilon(G)}{4r(G)} - \frac{(2\rho(G))^{\alpha-1}\xi_{AI}(G)^2}{2\xi_{\chi_\alpha}(G)} & if \alpha \le 1 \\ \frac{F^\epsilon(G)}{4r(G)} - \frac{(2r(G))^{\alpha-1}\xi_{AI}(G)^2}{2\xi_{\chi_\alpha}(G)} & if \alpha \ge 1 \end{cases}$$

and each equality is attained if and only if G *is self centered.*

Proof. Since $\frac{\epsilon_x^2+\epsilon_y^2}{2r(G)} \ge \frac{\epsilon_x^2+\epsilon_y^2}{\epsilon_x+\epsilon_y} = \frac{2\epsilon_x\epsilon_y}{\epsilon_x+\epsilon_y} + \frac{(\epsilon_x-\epsilon_y)^2}{\epsilon_x+\epsilon_y}$. Applying summation for $xy \in E(G)$ on both sides, we have

$$\sum_{xy\in E(G)} \frac{\epsilon_x^2 + \epsilon_y^2}{2r(G)} \ge \sum_{xy\in E(G)} \frac{2\epsilon_x\epsilon_y}{\epsilon_x + \epsilon_y} + \sum_{xy\in E(G)} \frac{(\epsilon_x - \epsilon_y)2}{\epsilon_x + \epsilon_y}$$

This implies that

$$\frac{\xi_F(G)}{2r(G)} \ge 2\xi_{ISI}(G) + \sum_{xy\in E(G)} \frac{(\epsilon_x - \epsilon_y)^2}{\epsilon_x + \epsilon_y}. \tag{2.8}$$

If $\alpha \geq 1$, then $\frac{(1-\alpha)}{2} \geq 0$ and Cauchy-Schwarz inequalities, we have

$$\begin{aligned}(2\rho(G))^{\alpha-1}\xi_{AI}(G)^2 = \frac{\xi_{AI}(G)}{(2\rho(G))^{\frac{1-\alpha}{2}}} &\leq \left(\sum_{xy\in E(G)} \frac{|\epsilon_x - \epsilon_y|}{(\epsilon_x + \epsilon_y)^{\frac{1-\alpha}{2}}}\right)^2 \\ &\leq \left(\sum_{xy\in E(G)} \frac{(\epsilon_x - \epsilon_y)^2}{\epsilon_x + \epsilon_y}\right)\left(\sum_{xy\in E(G)} (\epsilon_x + \epsilon_y)^{\alpha}\right) \\ &= \xi^{\alpha}_{\chi}(G) \sum_{xy\in E(G)} \frac{(\epsilon_x - \epsilon_y)^2}{\epsilon_x + \epsilon_y}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{\xi_F(G)}{2r(G)} &\geq 2\xi_{ISI}(G) + \frac{(2\rho(G))^{\alpha-1}\xi_{AI}(G)^2}{\xi_{\chi_\alpha}(G)} \\ \implies \xi_{ISI}(G) &\leq \frac{F^{\epsilon}(G)}{4r(G)} - \frac{(2\rho(G))^{\alpha-1}\xi_{AI}(G)^2}{2\xi_{\chi_\alpha}(G)} \\ (2r(G))^{\alpha-1}\xi_{AI}(G)^2 &\leq \xi_{\chi_\alpha}(G) \sum_{xy\in E(G)} \frac{(\epsilon_x - \epsilon_y)^2}{\epsilon_x + \epsilon_y}. \qquad (2.9)\end{aligned}$$

If $\alpha \geq 1$, then $\frac{(1-\alpha)}{2} \leq 0$ and $(2r(G))^{\alpha-1}\xi_{AI}(G)^2 \leq \chi_{\alpha}(G) \sum_{xy\in E(G)} \frac{(\epsilon_x-\epsilon_y)^2}{\epsilon_x+\epsilon_y}$.

Equation (2.8) implies that

$$\begin{aligned}\frac{\xi_F(G)}{2r(G)} &\geq 2\xi_{ISI}(G) + \frac{(2r(G))^{\alpha-1}\xi_{AI}(G)^2}{\xi_{\chi_\alpha}(G)} \\ \implies \xi_{ISI}(G) &\leq \frac{F^{\epsilon}(G)}{4r(G)} - \frac{(2r(G))^{\alpha-1}\xi_{AI}(G)^2}{2\xi_{\chi_\alpha}(G)}.\end{aligned}$$

This arguments gives the required result. If some bound is attained, then $\epsilon_x\epsilon_y = r^2(G)(G)$, $\forall uv \in E(G)$. Thus $\epsilon_x = r(G)$ $\forall u \in V(G)$ and G is self centered graph.

If G is self centered graph, then

$$\begin{aligned}\frac{F^{\epsilon}(G)}{4r(G)} - \frac{(2\rho(G))^{\alpha-1}\xi_{AI}(G)^2}{2\xi^{\alpha}_{\chi}(G)} &= \frac{F^{\epsilon}(G)}{4r(G)} - \frac{(2r(G))^{\alpha-1}\xi_{AI}(G)^2}{2\xi^{\alpha}_{\chi}(G)} \\ &= \frac{F^{\epsilon}(G)}{4r(G)} \\ &= \frac{2r^2(G)m}{4r(G)} \\ &= \frac{r(G)m}{2} \\ &= \xi_{ISI}(G).\end{aligned}$$

■

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