Codes from k-resolving sets for some Rook's graphs

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Abstract

A k-resolving set S is a set of vertices $\{v_1, v_2, ...v_l\}$ of a graph $G(V, E)$ if for distinct vertices $u, w \in V$, the lists of distances $(d_G(u, v_1), d_G(u, v_2), ..., d_G(u, v_l))$ and $(d_G(w, v_1), d_G(w, v_2), ..., d_G(w, v_l))$ differ in at least k-positions. The least size of a k-resolving set is called the k-metric basis of G and its cardinality is called the k-metric dimension, denoted by $dim_k(G)$. In this paper, we determine error-correcting codes for Rook's graphs, that is, the Cartesian product of any two complete graphs denoted as $K_n \square K_m$, using k-resolving sets. We have also constructed an infinite family of Rook's graph of k-dimension. Further, we have studied the k-metric dimension of $K_n \square K_m$. An explicit formula for $dim_k(K_n \square K_m)$ is determined for some particular cases and the codes arising from k-resolving sets of $K_n \square K_m$ are developed.

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1 *Introduction*

1.1 *Error-correcting codes*

Error-control codes are used in detecting and correcting transmission errors that occur across some noisy channel. When photographs are transmitted to Earth from deep space, error-control codes are used to guard against noise caused by lightning and other atmospheric interruptions. Correcting errors is even more important when transmitting encrypted data for security. In a secure cryptographic system, changing one bit in the ciphertext propagates many changes in the decrypted plaintext. Therefore, it is of utmost importance to detect and correct errors that occur when transmitting enciphered data, [1].

The study of error-control codes is called Coding Theory. Coding Theory emerged following the publication of Claude Shannon's seminal 1948 paper, "A mathematical theory of communication", [28]. Coding Theory being an area of discrete applied mathematics includes the study and discovery of various coding schemes. These schemes are used to increase the number of errors that can be corrected during data transmission, [1].

Coding theory has diverse applications from transmission of financial information across telephone lines, data transfer from one computer to another, to information transmission from a distant source such as a weather or communications satellite [1].

Formally, an error-correcting code (or simply a code) is a collection $\mathcal{C}$ of vectors, called codewords, of given length l over a fixed alphabet. The Hamming distance between two codewords $\mathbf{x} = (x_1, ..., x_l)$, $\mathbf{y} = (y_1, ..., y_l)$ is the number of positions where they differ, that is, $|\{i : x_i \neq y_i\}|$. The minimum distance of $\mathcal{C}$ is the least Hamming distance between any two distinct codewords; If the minimum distance is D, then the correction capability of $\mathcal{C}$ is $r = \lfloor (D-1)/2 \rfloor$. Suppose that a codeword $\mathbf{x}$ is transmitted via a noisy channel causes errors to appear. We seek to find codes with higher error correction capabilities so as to reduce decoding efforts. However, a decoding algorithm is always necessary to retrieve the original message. Any encoding and decoding procedure involves linear algebra, group theory and combinatorics see [7, 16, 19].

1.2 k-*metric dimension*

Metric dimension being a distance-based parameter is a very interesting topic in graph theory. It has gained widespread attention because of its range of applications in various areas including network discovery and verification [5], geographical routing protocols [25], combinatorial optimization [26], sensor network [18], robot navigation [22] and chemistry [8].

Metric dimension was introduced separately by Slater [27] in 1975 and by Harary and Melter [17] in 1976. Despite the fact it is an old concept, their study became much popular in recent decades. Several variations of metric dimension in graphs are nowadays more or less well known and studied. For more information see the survey by Kuziak and Yero [24].

The k-metric dimension problem was introduced by Estrada - Moreno et al. [13].It is nothing but an extension of the classical metric dimension. In their paper [13] the k-metric dimension of path graphs, cycle graphs and trees were studied. The k-metric dimension of lexicographic products of graphs, corona product of graphs, unicyclic graphs was studied by Estrada et al. [14, 15, 12]. Yero et al. [30] showed that the decision problem regarding whether the k-metric dimension of a graph does not exceed a positive integer is NP-complete, which also shows the NP-hardness of computing $dim_k(G)$ for any graph G. Further, an algorithm is provided to compute the k-metric dimension and k-metric basis of any tree. Corregidor et al. [9] determined some bounds for k for which a graph is k-metric dimensional. Klavžar et al. [23] studied the k-metric dimension of hierarchical product of graphs, splice and link product of graphs. An integer linear programming model for finding the k-metric dimension and k-metric basis for a graph was developed. Bailey et al. [2] studied the k-metric dimension of grid graphs, that is the Cartesian product of path graphs.

Bailey et al. [2] have started a new trend with the usage of graph parameters in developing codes. Bailey et al. [2] used the metric property of a graph in developing effective codes. In their interesting paper [2], the authors have used k-resolving sets arising in graphs and simple graph product namely the grid graphs. In [20, 21], the authors have extended this technique to Cartesian products of different graphs, not just paths as done by Bailey and Yero,[2]. In this paper we extend this to another class of Cartesian products, namely the Rook's graphs, which are the Cartesian products of any two complete graphs to obtain codes using k-resolving sets in these graphs. We develop decoding algorithms and analyze their complexity.

We first give some important definitions and results about k-resolving sets.

1.3 k-*resolving sets*

We consider finite, simple, connected, undirected graphs. The length of a shortest path between two vertices u and v of a graph G is denoted by

$d_G(u, v)$, [29].

Definition 1.1. *[17, 27] Let $G = (V, E)$ be a graph. Given a set $S = \{v_1, v_2, ..., v_d\} \subseteq V(G)$ and a vertex $u \in V(G)$, the vector $r(u|S) = (d_G(u, v_1), d_G(u, v_2), ..., d_G(u, v_d))$ is called the metric representation of u with respect to S. The set S is called a resolving set for G if the metric representations of all vertices of G are pairwise different. That is, every pair of distinct vertices of G, have their metric representations differ in at least one position.*

Definition 1.2. *[17, 27] A resolving set with the smallest possible cardinality is called a metric basis of G.*

Definition 1.3. *[17, 27] The cardinality of the metric basis of G is called metric dimension of G, denoted by $dim(G)$.*

Definition 1.4. *[13] Let $G = (V, E)$ be a graph. An ordered set of vertices $\{v_1, v_2, ..., v_l\}$ is a k-resolving set for G if, for any distinct vertices $u, w \in V$, the lists of distances $(d_G(u, v_1), d_G(u, v_2), ..., d_G(u, v_l))$ and $(d_G(w, v_1), d_G(w, v_2), ..., d_G(w, v_l))$ differ in at least k-positions.*

Definition 1.5. *[13] A k-resolving set with minimum cardinality is called a k-metric basis of G.*

Definition 1.6. *[13] The cardinality of a k-metric basis of G is called k-metric dimension of G, denoted by $dim_k(G)$.*

Definition 1.7. *[13] If k is the largest integer for which G has a k-resolving set, then G is called a k-metric dimensional graph.*

Definition 1.8. *[13] Given two vertices $x, y \in V(G)$, the set of distinctive vertices of x, y is denoted by $\mathcal{D}_G(x, y)$ and is defined as $\mathcal{D}_G(x, y) = \{z \in V(G) : d_G(x, z) \neq d_G(y, z)\}$.*

Many results were derived using this definition.

Theorem 1.9. *[13] A connected graph G is k-metric dimensional if and only if $k = min_{x,y \in V(G)} |\mathcal{D}_G(x, y)|$.*

Corollary 1.10. *[13] If a graph G is k-metric dimensional then $dim_k(G) \geq |\mathcal{D}_k(G)|$ where $\mathcal{D}_k(G) = \bigcup_{|\mathcal{D}_G(x,y)|=k} \mathcal{D}_G(x, y)$.*

Definition 1.11. *[29] The Cartesian product of G and H, written, $G \square H$, is the graph with vertex set $V(G) \times V(H)$ specified by putting (u, v) adjacent to (u', v') if and only if (i) $u = u'$ and $vv' \in E(H)$, or (ii) $v = v'$ and $uu' \in E(G)$.*

2 *The* $\mathbf{k}$*-metric dimension of* $\mathbf{K_n \square K_m}$

In this section, we will show, for any graph $K_n \square K_m$,$n \in \{3,4\}$ and $m \geq 7$ of order nm and any $k \in \{1,2,....,2n\}$,

$$dim_k(K_n \square K_m) = \begin{cases} m-1, & \text{if } k=1, \\ \left\lfloor \frac{2}{3}(m+n+1) \right\rfloor, & \text{if } k=2, \\ \dfrac{(k+1)}{2}m-1, & \text{if } k \geq 3 \text{ and } k \text{ is odd}, \\ \dfrac{km}{2}, & \text{if } k \geq 4 \text{ and } k \text{ is even}. \end{cases}$$

This goes on to provide an interesting infinite family of k- metric dimensional graphs namely Rook's graphs.

Let G be the graph $K_n \square K_m$, and $U = \{u_1, u_2, ..., u_n\}$ and $V = \{v_1, v_2, ..., v_m\}$ be the vertex sets of K_n and K_m respectively.

It is clear that, if G is a k-metric dimensional graph, then for every natural number $k' \leq k$, G also has a k'-metric basis. A characterization of k-metric dimensional graphs obtained in Theorem 1.9 [13], will be useful in our work.

To compute the k-metric dimension of Rook's graphs, we need to first determine for which values of k there exists a k-metric basis. This is answered by our first result.

Theorem 2.1. *The Rook's graph $G = K_n \square K_m$ is $2n$-metric dimensional for any integers $m, n \geq 3$ where $m \geq n$.*

Proof. We will first show that $k \leq 2n$. For this, we consider the vertices (u_1, v_1) and (u_1, v_2). Notice that,

$$\mathcal{D}_G\big((u_1, v_1), (u_1, v_2) \big) = U \times \{v_1, v_2\}.$$

Thus, $\left|\mathcal{D}_G\big((u_1, v_1), (u_1, v_2) \big)\right| = 2n$, and this is the largest minimum value possible (since we consider the full vertex set of G). Hence G is k-metric dimensional for some $k \leq 2n$. Next we will show that $k \geq 2n$. Consider any two distinct vertices of G namely, (u_i, v_j) and (u_s, v_t). We consider the following cases to determine the number of distinctive vertices.

Case 1. $i = s$.

Hence $j \neq t$ and it follows that

$$\{u_i\} \times \{v_j, v_t\} \subseteq \mathcal{D}_G\big((u_i, v_j), (u_s, v_t) \big)$$

for $i = 1, 2, ..., n$.
Also, $\big| \big(\{u_i\} \times V \big) \cap \mathcal{D}_G\big((u_i, v_j), (u_s, v_t) \big) \big| = 2$ for $i = 1, 2, ...n$.

Thus, we have

$$\begin{aligned} \big|\mathcal{D}_G\big((u_i, v_j), (u_s, v_t)\big)\big| &\geq |\{u_i\} \times \{v_j, v_t\}| + |(\{u_i\} \times V) \cap D_G\big((u_i, v_j), (u_s, v_t)\big)| \\ &\quad -2, \text{ for } i = 1, 2, ..., n. \\ &\geq 2 + 2 - 2 \geq 2, \text{ for } i = 1, 2, ..., n. \end{aligned}$$

Hence in general,

$$\begin{aligned} \big|\mathcal{D}_G\big((u_i, v_j), (u_s, v_t)\big)\big| &\geq 2 + 2 + 2 + ... + 2 \ (n \text{ times}) \\ &\geq 2n. \end{aligned}$$

Case 2. $j = t$.

Proof is similar to Case 1.

Case 3. $i \neq s$ and $j \neq t$.

We may assume that $i < s$.

Hence we have one of the following situations.

Case 3(i). $j < t$.

We note that at the most two vertices of $\big(\{u_1, ..., u_s\} \times \{v_j\} \big) \cup \big(\{u_i, ..., u_n\} \times \{v_t\} \big) \cup \big(\{u_i\} \times \{v_1, ..., v_t\} \big) \cup \big(\{u_s\} \times \{v_j, ..., v_m\} \big)$ do not belong to the set $\mathcal{D}_G\big((u_i, v_j), (u_s, v_t) \big)$.

Consequently,

$$\begin{aligned} \big|\mathcal{D}_G\big((u_i, v_j), (u_s, v_t) \big)\big| &\geq |\{u_1, ..., u_s\} \times \{v_j\}| + |\{u_i, ..., u_n\} \times \{v_t\}| \\ &\quad + |\{u_i\} \times \{v_1, ..., v_t\}| + |\{u_s\} \times \{v_j, ..., v_m\}| - 6. \\ &\geq s + (n - i + 1) + t + (m - j + 1) - 6 \\ &= s + t + m + n - i - j - 4 \\ &\geq i + j + m + n - i - j - 4, \ (\because i < s, j < t). \\ &\geq m + n - 4. \end{aligned}$$

We note that $m + n - 4 \geq n$ if $m \geq 4$ and is $\geq n - 1$ if $m = 3$. When $m \geq 4$, we have $m + n \geq n + 4$. Multiplying both sides by 2, we get, $2(m + n) \geq 2n + 8 \geq 2n$. When $m = 3$, we have $m + n \geq n + 3$. Multiplying both sides by 2, we get, $2(m + n) \geq 2n + 6 \geq 2n$. Thus, $\big|\mathcal{D}_G\big((u_i, v_j), (u_s, v_t) \big)\big| \geq 2n$.

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Case 3(ii). $j > t$.

Proof is same as $j < t$.

As a result of all the cases, we obtain $k \geq 2n$. Hence $k = 2n$. By Theorem 1.9 [13] a graph is k-metric dimensional if and only if $k = min_{x,y \in V(G)} |\mathcal{D}_G(x, y)|$. Hence $K_n \square K_m$ is $2n$-metric dimensional since $k = 2n$.

□

As in case of $P_s \square P_t$ [2], $K_n \square P_m$ [20] and $P_m \square C_n$ [21], the distinctive vertices play an important role here also. It is observed that for a pair x, y, if S is a k-resolving set for $G = K_n \square K_m$, then $|\mathcal{D}(x, y) \cap S| \geq k$. We use this fact in proving the existence of k-metric dimension for every $k \in \{1, 2, ..., 2n\}$ for some particular cases of $K_n \square K_m$. The result given below establishes this fact.

Theorem 2.2. *For the graph* $G = K_n \square K_m$*,*

$$dim_k(G) = \begin{cases} m - 1, & \text{if } k = 1, \\ \left\lfloor \dfrac{2}{3}(m + n + 1) \right\rfloor, & \text{if } k = 2, \\ \dfrac{(k+1)}{2} m - 1, & \text{if } k \geq 3 \text{ and } k \text{ is odd}, \\ \dfrac{km}{2}, & \text{if } k \geq 4 \text{ and } k \text{ is even}, \end{cases}$$

where n, m *and* k *are any integers such that* $n \in \{3, 4\}$, $m \geq 7$ *and* $k \in \{1, 2, ..., 2n\}$.

Proof. If $k = 1$, then from [6], we know that $\beta(K_n \square K_m) = m - 1$ if $m \geq 2n - 1$. Thus $dim_1(G) = dim(G) = m - 1$, so from now on, we consider only $k \geq 2$. Let (u_i, v_j) and (u_s, v_t) be two distinct vertices of G. The proof is divided into three cases depending on the value of k.

Case 1. k = 2.

We will first show that $dim_2(G) \leq \left\lfloor \frac{2}{3}(m + n + 1) \right\rfloor$. We consider the metric basis $S = \big(\{u_1\} \times \{v_1, v_2\}\big) \cup \big(\{u_2\} \times \{v_3, v_4\}\big) \cup \big(\{u_3\} \times \{v_5, v_6\}\big) \cup \big(\{u_n\} \times \{v_{m-1}, v_m\}\big)$. Consider the metric representations of each vertex with respect to the basis S (which can be obtained from the distance matrix of G). Clearly each vertex has a unique metric representation and each pair of vertices differ in atleast 2 positions. Thus, S is a 2-resolving set. From the definition of resolving sets, it follows that $dim_2(G) \leq \left\lfloor \frac{2}{3}(m + n + 1) \right\rfloor$. Next we show that $dim_2(G) \geq \left\lfloor \frac{2}{3}(m+n+1) \right\rfloor$. That is, we will prove that there exists no smaller 2-resolving set. We will prove this by method of contradiction. Suppose that there exists a 2-metric basis of cardinality less than $\left\lfloor \frac{2}{3}(m+n+1) \right\rfloor$, that is, $\left\lfloor \frac{2}{3}(m+n+1)-1 \right\rfloor$, denote it by S'. This means that $|\mathcal{D}_G(x,y) \cap S'| \geq 2$. Consider the metric representations of all the vertices of G with respect to the basis S'. For each pair x, y of metric representations, we determine the set of distinctive vertices $\mathcal{D}_G(x,y)$. We observe that $\mathcal{D}_G\big((u_1,v_1),(u_1,v_m)\big) = \{(u_1,v_1)\}$. Hence $|\mathcal{D}_G\big((u_1,v_1),(u_1,v_m)\big) \cap S'| \geq 1$ implies that S' is 1-metric basis, which is a contradiction to the fact that S' is a 2-metric basis. Hence, $|S'| \neq \left\lfloor \frac{2}{3}(m+n+1)-1 \right\rfloor$. Thus $|S'| \geq \left\lfloor \frac{2}{3}(m+n+1) \right\rfloor$. It follows that $dim_2(G) = \left\lfloor \frac{2}{3}(m+n+1) \right\rfloor$.

Case 2. k is odd and $k \geq 3$.

First we will prove that $dim_k(G) \leq \frac{(k+1)}{2}m - 1$. The metric basis for $k = 3, 5$ and 7 are as follows:

$$\begin{aligned} S_3 &= \big(\{u_1\} \times \{v_1, ..., v_{m-2}\}\big) \cup \big(\{u_2\} \times \{v_1, v_2, v_3, v_{m-1}, v_m\}\big) \\ &\quad \cup \big(\{u_3\} \times \{v_4, ..., v_{m-1}\}\big) \\ S_5 &= \big(\{u_1, u_2\} \times V\big) \cup \big(\{u_3\} \times \{v_1, v_2, ..., v_{m-1}\}\big) \\ S_7 &= \big(\{u_1, u_2, u_3\} \times V\big) \cup \big(\{u_4\} \times \{v_1, v_2, ..., v_{m-1}\}\big) \end{aligned}$$

Consider the metric representations of all vertices of G with respect to the above bases (this can be obtained from the distance matrix of G). It can easily be seen that for every $k = 3, 5$ and 7, each vertex has a unique representation and each pair of representations differs in atleast k positions, where $k = 3, 5$ and 7. Thus S_3 and S_5 and S_7 are 3-resolving, 5-resolving and 7-resolving respectively. Hence $dim_k(G) \leq \frac{(k+1)}{2}m - 1$.

Next we will show that $dim_k(G) \geq \frac{(k+1)}{2}m - 1$. We consider the following cases depending on m being odd or even,.

Case 2.1 m is odd.

Depending on n, we have the following cases.

Subcase 2.1.1 $n = 4$.

By Theorem 2.1, $k \leq 8$. In this case, we consider only odd k values namely $3, 5$ and 7. When $k = 7$, we have $dim_7(G) \leq 4m - 1$. We shall show that no

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smaller k-resolving set exists. We shall prove this by method of contradiction. Suppose S' is a k-metric basis of $G = K_n \square K_m$ with cardinality $< 4m-1$, that is, $4m-2$. Then,
$S' = \{(u_1,v_1),..,(u_1,v_m),(u_2,v_1),..,(u_2,v_m),(u_3,v_1),..,(u_3,v_m),(u_n,v_1),..,(u_n,v_{m-2})\}$.
We consider the vertices $\big(u_1,v_1\big), \big(u_2,v_1\big), \big(u_1,v_m\big)$ and $\big(u_n,v_m\big)$.
Let $A = \mathcal{D}_G\big((u_1,v_1),(u_2,v_1)\big) \cap S'$ and $B = \mathcal{D}_G\big((u_1,v_m),(u_n,v_m)\big) \cap S'$.
Now,
$|A| = \big|\mathcal{D}_G\big((u_1,v_1),(u_2,v_1)\big) \cap S'\big| = \big|\big(\{u_1,u_2\} \times V\big) \cap S'\big|$
$\quad \geq \dfrac{k(m+1)}{4} + (4s-4),$
and
$|B| = \big|\mathcal{D}_G\big((u_1,v_m),(u_n,v_m)\big) \cap S'\big| = \big|\big(\,\{u_1,u_n\} \times V\big) \cap S'\big|$
$\quad \geq \dfrac{k(m+1)}{4} + (4s-6),$
where $s \geq 1 \in \mathbb{N}$ depends on m. Depending on the value of m as m is odd we choose m has $m+1$ or $m-1$.

We have,$|S'| \geq |A| + |B|$

$$|S'| = \frac{k(m+1)}{2} + (8s-10). \text{ This is same as } \frac{k(m+1)}{2} - 1$$

Now suppose $|S'| = \dfrac{k(m+1)}{2} - 1$.Then, it follows that, $|A| = \dfrac{k(m+1)}{4} + (4s-4)$ and $|B| = \dfrac{k(m+1)}{4} + (4s-6)$. Consider $\big(u_2,v_m\big)\big(u_3,v_{m-1}\big) \in S$ and $S = A \cup B$. Let P_1 denote all the set of vertices of G except $\big(u_2,v_m\big)$ and P_2 denote all the set of vertices of G except $\big(u_3,v_{m-1}\big)$. That is, $P_1 = \big(\{u_1,u_3,u_n\} \times V\big) \cup \big(\{u_2\} \times \{v_1 ..., v_{m-1}\}\big)$ and $P_2 = \big(\{u_1,u_2,u_n\} \times V\big) \cup \big(\{u_3\} \times \{v_1,...,v_{m-2},v_m\}\big)$.
So,

$$|P_1 \cap S'| = \frac{k(m+1)}{4} + 4s - 5$$

and

$$|P_2 \cap S'| = \frac{k(m+1)}{4} + 4s - 7.$$

We now consider the vertices along the diagonal namely $(u_1,v_1),(u_2,v_2),...,(u_n,v_n)$. For each pair of these vertices, we denote the set of distinctive vertices by Q, that is, $Q = \mathcal{D}_G\big((u_1,v_1),(u_2,v_2)\big)$. We notice that since S' is a k-metric basis,
$|Q \cap S'| \geq \dfrac{k(m+1)}{4} + (4s-4)$ and $|Q \cap S'| \geq \dfrac{k(m+1)}{4} + (4s-6)$ for the

pairs of vertices along the diagonal.
Thus,

$$|Q \cap S'| \leq |P_1 \cap S'| = \frac{k(m+1)}{4} + 4s - 5,$$

and

$$|Q \cap S'| \leq |P_2 \cap S'| = \frac{k(m+1)}{4} + 4s - 7,$$

This is a contradiction to the fact that S' is a k-metric basis of cardinality $4m-2$. Hence $|S'| \neq 4m-2$. Thus, $|S'| \geq 4m-2+1 = 4m-1$.
Applying a similar process when $k = 5$ and 3, we arrive at similar contradictions. Therefore, $|S'| \neq \frac{k(m+1)}{2} - 1$. Since m is odd, $|S'| \geq \left(\frac{k+1}{2}\right)m - 1$.
Subcase 2.1.2 $n = 3$.
In this case, $k \leq 6$ (by Theorem 2.1). Thus, for odd k, we consider only $k = 3$ and 5. Rest of the proof is similar to Subcase 2.1.1.

Case 2.2 m is even.
Subcase 2.2.1 $n = 3$.
By Theorem 2.1, we know that $k \leq 6$. Thus, we consider only $k = 3$ and $k = 5$ for this case. When $k = 5$, we have $dim_5(G) \leq 3m - 1$. We shall show that no smaller k-resolving set exists. We shall prove this by method of contradiction. Suppose S' is a k-metric basis for $G = K_n \square K_m$ of cardinality $< 3m - 1$, that is, $3m - 2$.
Then, $S' = \{(u_1, v_1), .., (u_1, v_m), (u_2, v_1), .., (u_2, v_m), (u_n, v_1), .., (u_n, v_{m-2})\}$.
We consider the vertices (u_1, v_1), (u_2, v_1), (u_1, v_m) and (u_n, v_{m-2}). Let $A = \mathcal{D}_G\big((u_1, v_1), (u_2, v_1)\big) \cap S'$ and $B = \mathcal{D}_G\big((u_1, v_m), (u_n, v_{m-2})\big) \cap S'$.
Now,

$$|A| = \left|\mathcal{D}_G\big((u_1, v_1), (u_2, v_1)\big) \cap S'\right| = \left|\big(\{u_1, u_2\} \times V\big) \cap S'\right| \geq \left(\frac{k-1}{4}\right)m + m,$$

and

$$|B| = \left|\mathcal{D}_G\big((u_1, v_m), (u_n, v_{m-2})\big) \cap S'\right| = \left|\big(\{u_1, u_n\} \times V\big) \cap S'\right|$$
$$\geq \left(\frac{k-1}{4}\right)m - 2s, \text{ where } s \geq 1 \in \mathbb{N} \text{ depends on } m.$$

We have, $|S'| \geq |A| + |B|$

$$|S'| = \left(\frac{k-1}{2}\right)m + (m - 2s). \text{ This is same as } \left(\frac{k+1}{2}\right)m - 2.$$

Now suppose $|S'| = \left(\frac{k+1}{2}\right)m - 2$.Then, it follows that, $|A| = \left(\frac{k-1}{4}\right)m + m$ and $|B| = \left(\frac{k-1}{4}\right)m - 2s$. Consider $(u_2, v_m), (u_1, v_m) \in S$ and $S = A \cup B$.
Let P_1 denote all the set of vertices of G except (u_2, v_m) and P_2 denote all

the set of vertices of G except $\big(u_1, v_m\big)$. That is, $P_1 = \big(\{u_1, u_n\} \times V\big) \cup \big(\{u_2\} \times \{v_1 ..., v_{m-1}\}\big)$ and $P_2 = \big(\{u_2, u_n\} \times V\big) \cup \big(\{u_1\} \times \{v_1, ..., v_{m-1}\}\big)$.
So,

$$|P_1 \cap S'| = \left(\frac{k-1}{4}\right)m + m - 1$$

and

$$|P_2 \cap S'| = \left(\frac{k-1}{4}\right)m - 2s - 1.$$

We now consider the vertices along the diagonal namely $(u_1, v_1), (u_2, v_2), ..., (u_n, v_n)$. For each pair of these vertices, we denote the set of distinctive vertices by Q, that is, $Q = \mathcal{D}_G\big((u_1, v_1), (u_2, v_2)\big)$. We notice that since S' is a k-metric basis,
$|Q \cap S'| \geq \left(\frac{k-1}{4}\right)m + m$ and $|Q \cap S'| \geq \left(\frac{k-1}{4}\right)m - 2s$ for the pairs of vertices along the diagonal.
Thus,

$$|Q \cap S'| \leq |P_1 \cap S'| = \left(\frac{k-1}{4}\right)m + m - 1,$$

and

$$|Q \cap S'| \leq |P_2 \cap S'| = \left(\frac{k-1}{4}\right)m - 2s - 1.$$

This is a contradiction to the assumption that S' is a k-metric basis of cardinality $3m-2$. Hence $|S'| \neq 3m-2$. Thus, $|S'| \geq 3m-2+1 = 3m-1$. Applying a similar process when $k=3$, we arrive at similar contradiction. Therefore, $|S'| \neq \left(\frac{k+1}{2}\right)m - 2$. Hence, $|S'| \geq \left(\frac{k+1}{2}\right)m - 1$.

Subcase 2.2.2 $n = 4$.
By Theorem 2.1, $k \leq 8$. In this case, we consider $k = 3, 5$ and 7. Proof is similar to Subcase 2.2.1.
As a consequences of all the above cases, $dim_k(G) \geq \left(\frac{k+1}{2}\right)m - 1$.
Hence $dim_k(G) = \left(\frac{k+1}{2}\right)m - 1$.

Case 3 k is even and $k \geq 4$.
In this case, the metric basis for $k = 4, 6$ and 8 are as follows:

$$S_4 = \big(\{u_1\} \times \{v_1, ..., v_{m-2}\}\big) \cup \big(\{u_2\} \times \{v_1, v_2, v_3, v_{m-1}, v_m\}\big)$$

$$\cup\big(\{u_3\} \times \{v_4, ..., v_m\}\big)$$
$$S_6 = \big(\{u_1, u_2, u_3\} \times V\big)$$
$$S_8 = \big(\{u_1, u_2, u_3, u_4\} \times V\big)$$

Consider the metric representations of all vertices of G with respect to the above bases. It can easily be seen for each $k = 4, 6$ and 8, each vertex has a unique representation and each pair of representations differs in atleast k positions, where $k = 4, 6$ and 8. Thus S_4 and S_6 and S_8 are 4-resolving, 6-resolving and 8-resolving respectively. Hence $dim_k(G) \leq \frac{km}{2}$.

Next we will show that $dim_k(G) \geq \frac{km}{2}$. Depending on n, we have the following cases.

Subcase 3.1 $n = 4$.
By Theorem 2.1, $k \leq 8$. In this case, we consider only even k values namely $4, 6$ and 8. When $k = 8$, we have $dim_8(G) \leq 4m$. In this case, $\mathcal{D}_8(G) = \bigcup_{|\mathcal{D}(x,y)|=8} \mathcal{D}(x, y) = V(K_4 \square K_m) = 4m$. By Theorem 2.1, we know that $K_4 \square K_m$ is 8-metric dimensional. Thus by Corollary 1.11, it follows that, $dim_8(G) \geq |\mathcal{D}_8(G)| = 4m$. Hence $dim_8(G) = 4m$. When $k = 4$ and 6, we follow the same procedure as when k is odd and arrive at similar contradictions. Therefore, $|S'| \neq \frac{km}{2} - 1$. Thus, $|S'| \geq \frac{km}{2}$.

Subcase 3.2 $n = 3$.
By Theorem 2.1, we know that $k \leq 6$. Thus, we consider only $k = 4$ and $k = 6$ for this case. Proof is similar to Subcase 3.1.

As a consequence of all the subcases, $dim_k(G) \geq \frac{km}{2}$.
Thus, $dim_k(G) = \frac{km}{2}$. Hence the proof.

□

3 *Codes from Rook's graph* $\mathbf{K_n \square K_m}$

In this section, we use the family of Rook's graphs, that is, the Cartesian product of the complete graph K_n with complete graph K_m, to obtain $(K_n \square K_m, k)$-codes.

The interesting application of purely graph theoretic concept of k-resolving sets comes in the form of generating codes from these. These two were nicely merged by Bailey and Yero [2], wherein they derived codes from k-resolving sets. For ready reference we list some important definitions and results that help us in getting our main results.

Definition 3.1. *[2] Let G be a graph with n vertices and diameter d, and let $S = \{v_1, v_2, ..., v_l\}$ be a k-resolving set for G of size l. Then the set $\mathcal{C}(G, S) = \Big\{\big(d_G(u, v_1), d_G(u, v_2), ..., d_G(u, v_l)\big) : u \in V\Big\}$ is called a (G, k)-code.*

Remark 3.2.

1. *$\mathcal{C}(G, S)$ is an error-correcting code of length l, size n and with minimum hamming distance at least k, over the alphabet $\{0, ..., d\}$, which can correct $r = \lfloor (k-1)/2 \rfloor$ errors. Also $k \geq 3$ for $r > 0$.*

2. *For $\mathcal{C}(G, S)$ to be used for error correction, we need a decoding algorithm see section 3.1.*

3. *Uncoverings was introduced by Bailey in [3, 4] where they were applied to decoding permutation codes.*
   Uncovering design is defined as [2] : Let ν,κ,τ be integers such that $\nu \geq \kappa \geq \tau \geq 0$. A $(\nu, \nu - \kappa, \tau)$-uncovering is a collection $\mathcal{U}$ of $(\nu - \kappa)$-subsets of $\{1, ..., \nu\}$ with the property that any τ-subset of $\{1, ..., \nu\}$ is disjoint from at least one member of $\mathcal{U}$. Taking the complements of each $(\nu - \kappa)$-subset in $\mathcal{U}$, we obtain a (ν, κ, τ)-covering design.
   We use covering and uncovering designs in our decoding algorithm.

The following result gives the codewords obtained from k-resolving sets of $K_n \square K_m$.

Theorem 3.3. *A $(K_n \square K_m, k)$-code has nm codewords of length*

$$l = dim_k(K_n \square K_m) = \begin{cases} m - 1, & \text{if } k = 1, \\ \left\lfloor \frac{2}{3}(m + n + 1) \right\rfloor, & \text{if } k = 2, \\ \dfrac{(k+1)}{2} m - 1, & \text{if } k \geq 3 \text{ and } k \text{ is odd}, \\ \dfrac{km}{2}, & \text{if } k \geq 4 \text{ and } k \text{ is even}, \end{cases}$$

over an alphabet of size $diam\,(K_n \square K_m) = 2$ with the minimum distance k and so can correct $r = \lfloor (k-1)/2 \rfloor$ errors.

Proof. Proof follows from the above two Theorems 2.1 and 2.2 respectively. □

We illustrate using the following example.

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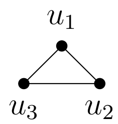
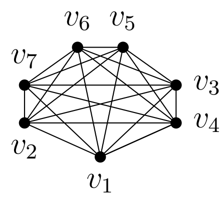
Example 3.4. *Consider $K_3 \square K_7$ as in Figure 1.*
By Theorem 2.1, we know that $K_3 \square K_7$ is 6-metric dimensional.
From Theorem 2.2, $dim_6(G) = \dfrac{6 \times 7}{2} = 21$.
Thus, $S = V(K_3 \square K_7)$ is a 6-resolving set.

K_3 :



K_7 :



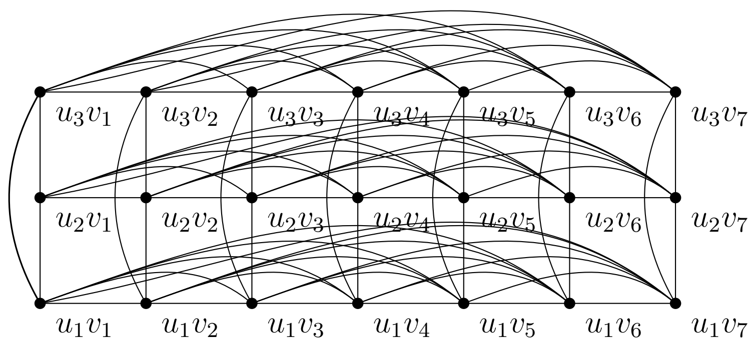


Figure 1: $K_3 \square K_7$

We find the distance of each vertex in S with every vertex of $K_3 \square K_7$ which is nothing but its distance matrix, as shown below. Note that each row

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represents the distance of one vertex to all the vertices of $K_3\square K_7$.

$$\begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \\
1 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 1 & 2 & 2 & 2 & 2 & 2 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 2 & 2 & 2 & 2 & 1 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \\
1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}.$$

Next we determine its minimum hammming distance. The hamming distance between each pair of vertices is given in matrix form below.

$$\begin{bmatrix}
6 & 6 & 6 & 6 & 6 & 6 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 \\
6 & 6 & 6 & 6 & 6 & 6 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 \\
6 & 6 & 6 & 6 & 6 & 6 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 \\
6 & 6 & 6 & 6 & 6 & 6 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 \\
6 & 6 & 6 & 6 & 6 & 6 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 \\
6 & 6 & 6 & 6 & 6 & 6 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 \\
6 & 6 & 6 & 6 & 6 & 6 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 \\
14 & 14 & 14 & 14 & 14 & 14 & 6 & 6 & 6 & 6 & 6 & 6 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 \\
14 & 14 & 14 & 14 & 14 & 14 & 6 & 6 & 6 & 6 & 6 & 6 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 \\
14 & 14 & 14 & 14 & 14 & 14 & 6 & 6 & 6 & 6 & 6 & 6 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 \\
14 & 14 & 14 & 14 & 14 & 14 & 6 & 6 & 6 & 6 & 6 & 6 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 \\
14 & 14 & 14 & 14 & 14 & 14 & 6 & 6 & 6 & 6 & 6 & 6 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 \\
14 & 14 & 14 & 14 & 14 & 14 & 6 & 6 & 6 & 6 & 6 & 6 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 \\
14 & 14 & 14 & 14 & 14 & 14 & 6 & 6 & 6 & 6 & 6 & 6 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 \\
14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6
\end{bmatrix}.$$

We observe that the minimum hamming distance is 6.
Hence by Theorem 3.3*, a* $(K_3\square K_7, 6)$*-code has* 21 *codewords over the alphabet* $\{0,1,2\}$ *and has minimum distance* 6*, so can correct* $r = \left\lfloor \dfrac{6-1}{2} \right\rfloor = 2$ *errors.*

The set of codewords obtained from distance matrix is as follows:
$\{011111112222221222222, 101111121222222122222, 110111122122222212222, 111011122212222221222, 111101122221222222122, 111110122222122222212, 111111022222212222221, 122222201111111222222, 212222210111112122222,$

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2212222110111112212222, 2221222111011112221222, 2222122111110112222122,
2222212111111012222212, 2222221111111102222221, 12222221222220111111,
21222222122222211011111, 22122222212222221101111, 222122222212221110111,
22221222222122211111011, 22222122222221211111101, 2222221222222111111110}.

In the next section, we obtain the decoding algorithm for $(K_n \square P_m, k)$-codes.

3.1 *Decoding Algorithm*

In this section we give a decoding algorithm which is a modification of the decoding algorithm given by Bailey and Yero [2].

Our decoding algorithm is as follows: Suppose we have received the word $x = x_1, x_2, ..., x_{dim_k(K_n \square K_m)}$ of length $dim_k(K_n \square K_m)$. We wish to correct r errors. Consider a $(K_n \square K_m, k)$-code having nm codewords of length $l = dim_k(K_n \square K_m)$ with error-correction capability $r = \lfloor (k-1)/2 \rfloor$. Let $r^{'} = 1 < r$. We determine the covering design $(dim_k(K_n \square K_m), r+1, r^{'} = 1)$. To obtain this, we partition $dim_k(K_n \square K_m) = a(r+1) + b$ into p subsets including as many as possible sets of size t where $t = \lfloor r + 1/r^{'} + 1 \rfloor = \lfloor r + 1/2 \rfloor$ and $p = \lceil v/t \rceil$. For any combination $r^{'}$ of the sets in the partition $dim_k(K_n \square K_m)$, take $r+1$ subsets of points which contain their union. Further, these $r+1$ subsets of points are chosen in such a way that $r+1$ subsets of points are disjoint from all other $r+1$ subsets of points(as much as possible). These form the blocks of $(dim_k(K_n \square K_m), r+1, r^{'} = 1)$. Taking the complement of these blocks we get $(dim_k(K_n \square K_m), dim_k(K_n \square K_m) - r - 1, r^{'} = 1)$-uncovering design.
Next we partition the received word $x = x_1, x_2, ..., x_{dim_k(K_n \square K_m)}$ and the blocks of the uncovering design into $r+1$ sets. We now compare each of $r+1$ bit of the received word with the distance matrix of $(K_n \square K_m, k)$ at positions indexed by all the blocks of the uncovering design such that atleast r elements must be identical to the elements of x. The row belonging to $(K_n \square K_m, k)$ having most number of common entries from all the blocks of the uncovering design is the original transmitted word. Thus on comparing the original received word with the common row (that we have obtained) we will obtain the error positions, which can easily be rectified.

Remark 3.5. *Our decoding is better than the decoding proposed by Bailey*

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and Yero [2]. In our decoding we fix $r' = 1$. *We consider covering designs* $(dim_k(K_n \square K_m), r+1, r')$ *instead of* $(dim_k(K_n \square K_m), r, r')$ *where* $r' < r$. *Also we consider uncovering designs* $(dim_k(K_n \square K_m), dim_k(K_n \square K_m) - r - 1, r')$ *instead of* $(dim_k(K_n \square K_m), dim_k(K_n \square K_m) - r, r')$. *We observe that in our decoding the number of blocks of uncovering design reduces. Further instead of checking each column corresponding to each symbol of the received word, we check only* $r+1$ *words such that atleast* r *words match with the codeword. This reduces the computation time.*

3.2 *Complexity*

Let $\mathcal{U}$ denote the uncovering design $(dim_k(K_n \square K_m), dim_k(K_n \square K_m) - r - 1, r')$ and let the total number of blocks of $\mathcal{U}$ be denoted by T. For each block of $\mathcal{U}$, the distance matrix of $K_n \square K_m$ is examined at most $|T| - 1$ times and this repeats for every block of $\mathcal{U}$. Since we have T blocks, it follows that the complexity of the decoding algorithm is $O(|T| - 1 \cdot |T|)$.

4 *Conclusion*

The k-metric dimension is an extension of the classical metric dimension. While determining the k-metric dimension of any arbitrary graph is NP-hard, we have tried to solve the open case for Rook's graphs for some particular cases.
In this paper we have obtained the k-metric dimension of Rook's graph. A general formula for $dim_k(K_n \square K_m)$ for some particular cases is obtained and the codes arising from k-resolving sets of $K_n \square K_m$ are developed. Decoding in terms of covering and uncovering designs are given. In future communications this procedure is extended to obtain codes from other (Cartesian) products of some standard classes of graphs such as stacked Prism graphs and $K_n \square P_m$.

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