A Short Discussion on Delay Differential Equations

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Abstract

In this article, we discuss about delay differential equations and their types and also we talk over their applications in terms of science and engineering. As we know mathematical models play an important role in time-delay systems, so we discuss about the predictions in mathematical models which are used for fundamental science.

Keywords: Delay Differential Equations (DDEs) or Functional Differential Equations (FDEs), Applications of Delay Differential Equations (DDEs).

1. Introduction:

In many real-world systems, the time delay is a naturally occurring process. Or in the mathematical terms, simply we can say delay indicates the values of the function at a previous time. Delay is time lag, natural process and also a man-made process. In several branches of science like physics, biology, geology, economics, and also in mathematics the appearance of delay differentiation is clearly exists. A lucid example of time delay in nature is called reforestation. When forests are destroyed by humans through cutting trees, this action will be done in a short interval of time or when the forests are destroyed because of natural catastrophes such as fires, hurricanes and floods, and in a short time the forests deceases. Forest destruction takes short time, but it might take at least 20 years of cultivation and planting to give life back to forest i.e. reaching any kind of maturity. e.g. redwoods and another example occur due to fact that animals must take time to digest their food before activities [4]. So, the time delay is a vital component of any dynamic process in life sciences. Now a days the best example for explaining delay is covid19 i.e. coronavirus, because if we delays in isolation then it obviously delay in treatment, and the situation can be atrocious.

As, we know that ordinary differential equations (ODEs) and partial differential equations (PDEs), used a that type of mathematical model which does not incorporate a dependence on past history. But the model incorporates a past history is categorized as delay differential equations (DDEs) or functional differential equations (FDEs). Here we will discuss about origin of DDEs in section 2, Application of delay differential equations in section 3, Application of delay differential equations in fundamental science in section 4, conclusion in section 5 as follows:

2. Origin of Delay Differential Equations:

Delay differential equations, differential integral equations, and functional differential equations have been studied for at least 200 years in 1911 with properties of linear equations discussed by Yang Kuang [4]. The progress of human learning and dependence on automatic control systems after World war it gave birth to a different type of equation named delay differential equation (DDEs). The historical development of DDEs in population dynamics dates back to the 1920s, when Volterra in 1927, investigated the predator-prey models, but unfortunately, momentum on studying DDEs did not pick up until half a century later. The new classical book of Belman and Cooke (1963) [1] is now credited for the study of DDEs. Afterward, Hale pushed the study of these models to a present level of depth. DDEs are a type of functional differential equation that takes into interpretation of the effect of past time. Therefore, DDEs are a better way to define natural phenomena and time-related problems [2].

During the last 60 years the theory of functional differential equations has been developed extensively and has become part of the vocabulary of researchers dealing with specific applications such as mechanics, nuclear reactors, and also distributed networks, heat flow, learning models, physiology, biology, etc. as well as many others. For this, we have to take a reference of Kolmanovaski. In these stochastic effects are also being considered but the theory is not as well developed. The stability of main theory of basic DDEs was developed or elaborated by pontrygin in 1942, however, after world-war II; there was a rapid growth of theory and its applications.

In 1950s, there was manageable action in a subject that led to the vital publications by Myshkis in 1955, Krasovskii in 1959, A new book of Bellman and Cooke in 1963 is now accredited for the study of Delay differential equations (DDEs). Subsequently Halanay (in 1966), Norkin (in 1971), Hale (in 1977), Yanushevski (in 1978), Marshal (in 1979) these variety of researches with publications lasted until his day in a change of domains, consequently these crucial research on functional differential equations (FDE) deals with linear equations and preservation of stability or instability of equilibrium under small nonlinear perturbations when the linearization was stable or unstable [1].

In year 1963 Bellman and Cooke introduced differential-difference equations with some features of theoretical aspect and role of these equations in different fields. They also discussed about first order linear differential difference equations (LDDEs) of neutral and advanced type with constant coefficients with their asymptotic behavior [1].

After that in year 1982 K.L. Cooke and Z. Grossman described about Discrete Delay, Distributed Delay and Stability Switches [2], In 1992 the stability of DDEs and oscillation of DDEs have discussed by K. Gopalsamy [3], In year 1993 DDEs with their applications in population have discussed by Yang Kuang [4], J.K. hale introduced Functional differential equations (FDEs) in year 1993 [5], Andrei Lebedev and Mikhail Belousov also discussed about FDEs [6], Baker et. al. describe modeling and analysis of time logs in some basic patterns in year 1998 [7]. The qualitative and quantitative role of delay that plays in basic time-lag models which proposed in population dynamics, epidemiology, physiology, immunology, neural networks and cell kinetics discussed by G. A. Bocharov and F. A. Rihan [8]. One of the best book written by T. Faria, P. Freitas. (2001) entitled topics in Functional Differential and Difference Equations (FDDEs) describe the applications of state dependent delay differential equation in population dynamics with its modeling and analysis [9]. J.K. Hale introduced about History of Delay Equations [10], The general theory for sensitivity analysis using mathematical models which contain time-lags and they used adjoint equations with direct methods to calculate the sensitivity functions while the parameters performing in the model are not any constants but variables of time discussed by F. A. Rihan [11]. A model prepared by using delay differential equations for tumor growth by M. Villasana and A. Radunskaya [12], Leon Glass and Michael Mackey (2009) developed Mackey-Glass equation[13], F. A. Rihan et. al. Also discussed a model of Delay differential for tumor-immune response, chemo immunotherapy and optimal control in year 2014 [14], S. Lakshmanan et. al. developed another model called the differential genetic regulatory networks model using time delays and Markovian jumping parameters for checking its stability analysis [15], R. Rakkiyappan et. al. also checked the stability of memristor-based complex-valued recurrent neural networks using time delays [16], F. A. Rihan et. al., discussed about an Inverse problem for delay differential equations [17], Hradyesh Kumar Mishra and Rajnee Tripathi solved delay differential equations by using Homotopy perturbation method via Laplace transform [18], Cimen, E., and Uncu, S. Solved of Delay Differential Equation via Laplace Transform [19].

2.1. Delay Differential Equations:

A delay differential equation is a generalization of the ordinary differential equation, which is suitable for a physical system that depends on past history. So, Delay differential equations are a type of differential equation in which the derivative of the unknown function at previous times. DDEs are also called time-delay systems, systems with downtime, or reverberation hereditary systems, equations with decamping argument, or differential-difference equations [4].

A delay differential equation is of the form:

$$\dot{x}(t) = f(t,x(t), x(t-\tau_1(t,x(t)), x(t-\tau_2)(t,x(t)) \ldots\ldots\ldots\ldots)$$

The delays τ_i, $i = 1, 2\ldots$ are measurable physical quantities and may be constant and a function of t and x itself. The function $\tau_i(t, x(t)) \geq 0$ is called the state-dependent, if it depends on value of x(t).

Also, a Delay Differential Equation (DDE) is a differential equation where the state variable appears with delayed argument and these can be divided into different types. For example, some problems require a complete history of solution and some requires only a limited history of solution [4].

2.2. Classification of Delay Differential Equations (DDEs):

Delays having several types defined as follows [2]:

Definition 2.1: DDEs with Constant Delay: A delay differential equation described as follows.

$\dot{x}(t) = f(t, x(t), x(t - \tau)), x(t) \in R^d$, where delay $\tau > 0$ is constant.

Definition 2.2: DDEs with Time-Dependent Delay: This delay is a type of variable delay.

$\dot{x}(t) = f(t, x(t), x(t - \tau(t)))$, $x(t) \in R^d$ where delay $\tau(t) > 0$ is a given function.

Definition 2.3: DDEs with State-Dependent Delay: This delay is a type of state-dependent delay.

$\dot{x}(t) = f(t, x(t), x(t - \tau(t, x(t))))$, $x(t) \in R^d$ where delay $\tau(t, x(t)) > 0$ depends on solution.

The basis on these types of delay we classify the delay differential equations like linear or non-linear delay differential equation, retarded delay differential equation, neutral delay differential equation, autonomous or non-autonomous delay differential equation (invariant under the change $t \rightarrow t+T$ for all $t \epsilon R$), and stochastic delay differential equation [2].

2.3 Small Delays can have large effects:

As we know that the delayed logistic equation with a discrete delay a large delay will destabilize its positive steady state, while a small delay will not affect the behavior of solutions. In Reality, researchers are ignoring delays, when they think that delays are small. But it is not true without our qualifications [4].

For example, absolutely invariant systems were studied by Schipanov and absolutely invariant, the system loses its property, when one takes a small delay into it.

We know that, delayed logistic equation is of the form [4],

$$\dot{x}(t) = rx(t)\,[1 - x(t-\tau) / K],$$

Here, we see that critical value that destabilizes the local stability of steady-state at $x(t) \equiv K$,

$$\tau_0 = \frac{\pi}{2r}$$

Because, for a large value of r, τ_0 will be small. So from this example, we have seen that if we neglected small delays the small is really small in each individual equation [1].

3. Application of Modeling in Delay:

Models of differential equations with delay have pervaded many scientific and technical fields in the last decades. The use of delay differential equations and partial delay differential equations to model problems with the presence of lags or hereditary effects has demonstrated a valuable balance between pragmatism and docility. Of special interest in recent years is the development and analysis of models with interactions between delay and random effects, through the use of stochastic and random delay differential equations. Indeed, we have contributions dealing with the construction of exact solutions, numerical methods, dynamical properties, and applications to mathematical modeling of phenomena and processes in biology, economics, and engineering, in both deterministic and stochastic settings. V. Gennadii, Demidenko and Inessa I. Matveeva described Solutions and Asymptotic Stability of class of second-order delay differential equations [21]. One of the best book written by F.A. Rihan (2021) is Delay Differential Equations and Applications to Biology, discussed about the application part of delay differential equations in biology with crucial role of delay in our daily life [22]. Santosh Ruhil and Muslim Mali [24] described an inverse problem An inverse problem in place of an abstract impulsive differential equation using constant delay, Also considered the parameter with an over-determined form on a minor solution.

Delay differential equations are also used in mathematical modeling like retarded delay differential equation has an area of application in radiation dumping, stochastic delay differential equation has an area of application in immune response [14] and blood cell production and distributed delay differential equation has an area of application in a model of HIV infection. It is mainly used for analysis and predictions in a different type of areas of life sciences, for example, population dynamics, epidemiology, immunology, physiology, and neural networks [14]. There are so many acute physiological diseases where the initial symptoms are manifested by an alteration or irregularity in a control system that is normally periodic, or by the onset of an oscillation in a non-oscillatory process. Such physiological diseases have been termed dynamical diseases by Glass and Mackey (1979) [15], who have made a systematic study of several important and interesting physiological models with time delays i.e. Delay models in physiology**.**

Now, the single fixed delay is used as an application in immunology and multiple fixed delays are used as an application in cancer chemotherapy and varying delay which is time-dependent and also a state-dependent has an application in transport delays and combustion in the chamber of a Turbojet engine. So to understand the disease behavior there are many mathematical models which are including complex and simple mathematical models like the SIR model used for covid- 19 [23].

4. Applications of Delay Differential Equations in Fundamental Science:

Fundamental science is the basis of multidisciplinary action and applied science and we know formal science, natural science, social science are consist of fundamental science. There is simplest and best examples for use of delay differential equations in fundamental science are climate change system and coronavirus. Because for formal science like logic, the mathematics we can predict that change in the climate system and by the using of the mathematical model we can also predicts the delay by covid 19 [23] and delay in everything and socially now a days it can be easily seen that there is climate change in all over the world and also we know that covid 19 or coronavirus still impact on our social life and also delays. And in natural science it has also an impact on environment, earth's atmosphere etc., and also coronavirus is spreading due to the bad impact of nature [20].

Climate change is a natural phenomenon and takes place over long centuries. As we know climate change is one of the major challenge for whole over the world. Climate is the average weather for a region over a long period of time, affected by conditions and components of atmosphere. However there is no realistic approach to this climate problem. Global warming and Greenhouse gases are two responsible factors for climate change. Many of the authors used a conceptual climate model that describes the study of delayed feedback system with delay differential equations.

Furthermore, Coronavirus disease is an infectious disease caused by the SARS-Cov-2 virus. For covid-19 we have delayed mathematical models known as epidemic models like SIR, SEIR, model and many more that are used to construct describe the spread of infection. For established a type of infectious disease model based on a time delay dynamics system is SIR [23], model analyses the dynamics of infectious disease with a concurrent spread of disease awareness. The steady-state of the SEIR model is determined to control the pandemic [20].

5. Conclusion:

This paper reviews about that how delay differential equations came in to existence and also types of delays into the system; namely, constant delay, time dependent delay and state dependent delay. An alternative of delays depends on a system required. It tells us about some applications in mathematical modeling and some applications in fundamental science like climate change have a conceptual climate model and covid-19 has a delayed mathematical model.

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