Delay-dependent LMI condition to the stability of a category of uncertain discrete-time systems exerting quantization/overflow nonlinearities and variable time-lags

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Abstract

This paper deals with the stability of nonlinear uncertain discrete-time systems with variable time-lags. The system under assumption involves norm-bounded parameter uncertainties, quantization/overflow nonlinearities and variable time-lags. A Lyapunov function based delay-dependent sufficient stability condition is derived by using a more relaxed finite-sum inequality. As a special case, a stability condition for constant delay is also derived in this paper. Finally, numerical examples are given to show the efficacy of the proposed conditions.

Keywords

Finite wordlength nonlinearity, Linear matrix inequality, Lyapunov method, Norm-bounded parameter uncertainty, Variable time-lag.

1. Introduction

While implementing the category of discrete-time systems via specific binary hardware, there is a chance of occurrence of nonlinearities for example quantization and/or overflow in the designed system. Due to such nonlinear effects, the unpredictability in the system occurs in the form of limit cycles or unwanted oscillations that may tend the system to be unstable. So, it is a key issue to find the range of system parameters under which the designed system is not affected by limit cycles. During the past, several studies available on the composite effects of quantization and overflow [1-6] for discrete realizable systems.

Another source of instability in the designed system is parameter uncertainty which arises due to several reasons like finite resolution of measuring equipment, variation in system parameters, modeling error or some ignored factor. The effects of parameter uncertainties are studied extensively in existing literature [3-18].

Besides nonlinearities and parameter uncertainties, another source of instability in discrete-time systems is time-lag. Time-lag occurs due to finite computational time or transportation lag required for the transmission of information among the various parts of the system. The generated time-lag in processing of information may be constant or varying with time. Available stability conditions in literature are classified into two different categories in which one is delay-independent [3, 10] and other is delay-dependent [4, 5, 6, 8, 13, 15, 17-32]. Usually, delay-dependent conditions bring less conservative results because these conditions utilize the information of the size of the lag whereas delay-independent conditions have not included the information of the size of the lag. For expecting less conservative results, selection of appropriate Lyapunov function and application of tighter bounding techniques in the sum and cross terms of the forward difference of the Lyapunov function are the key steps. Various tighter bounding techniques from the existing literature are free-weighting matrix method [20, 33], Jensen inequality [21], Reciprocal convex method [6, 24], Wirtinger-based inequality [6, 17, 34] and Abel lemma-based inequality [35].

Practical engineering systems such as network control systems [14], discrete-time Markovian jump systems [22], discrete-time neural networks [2], Sensor networks [36] etc. are frequently suffered by the presence of above discussed instability factors. The problem of stability investigation of a category of discrete-time systems exerting quantization and/or overflow nonlinearities, parameter uncertainties and variable time-lags is more interesting and realistic in nature. Many results [3-6, 10, 17] have previously reported for the global asymptotic stability of these systems, yet there is a scope to achieve improved results over the previous results.

The key objective of the paper is to derive less conservative delay-dependent conditions for the global asymptotic stability of a category of uncertain discrete-time systems employing quantization and/or overflow nonlinearities and variable time-lags. The main contributions of this work are as follows:

1. A new global asymptotic stability criterion for uncertain discrete-time systems having quantization and/or overflow nonlinearities and variable time-lag is derived. The criterion utilizes a more relaxed technique [26, 28] which may reduce the conservatism and simplify the system analysis/synthesis process. As a special case of our main result, a stability condition for the category of uncertain discrete-time systems with constant time-lag and quantization/overflow nonlinearities is established.
2. The proposed conditions are in the setting of linear matrix inequalities (LMIs), and hence, one can easily test the conditions by using well-known LMI solvers [37, 39].
3. A comparison of the proposed results with the existing results [4-6] is given.

The remaining paper is outlined as follows. In Section 2, the system description and some useful lemmas are provided while in Section 3, the main results of the paper are derived. In Section 4, two numerical examples to demonstrate the effectiveness of the main results are given. Section 5 provides the conclusion of the paper and in Section 6, the future scope of the presented work is highlighted.

Notations: $\mathbf{0}$ is the null matrix or null vector; $\boldsymbol{I}$ is the identity matrix; $\boldsymbol{R}^{\alpha\times\beta}$ is the set of $\alpha\times\beta$ real matrices; $\boldsymbol{R}^{\alpha}$ is the set of $\alpha\times 1$ real vectors; $\boldsymbol{M}^T$ denotes the transpose of matrix $\boldsymbol{M}$; $\boldsymbol{M}>\mathbf{0}$ ($\boldsymbol{M}<\mathbf{0}$) means that $\boldsymbol{M}$ is positive (negative) definite symmetric matrix; maximum and minimum eigen values of a matrix $\boldsymbol{M}$ are $\lambda_{\max}(\boldsymbol{M})$ and $\lambda_{\min}(\boldsymbol{M})$, respectively; *diag* ($z_1, z_2,\ldots, z_g$) is a diagonal matrix with diagonal entries $z_1, z_2,\ldots, z_g$; the symmetric entries of a symmetric matrix is inferred by $*$.

2 System Description

The system under inspection is as

$$\chi(u+1)=\boldsymbol{O}\{\boldsymbol{Q}(\gamma(u))\}=\boldsymbol{\lambda}(\gamma(u))$$

$$=\left[\lambda_1(\gamma_1(u))\ \ \lambda_2(\gamma_2(u))\cdots\lambda_n(\gamma_n(u))\right]^T, \tag{1a}$$

$$\gamma(u)=(\boldsymbol{A}+\Delta\boldsymbol{A})\chi(u)+(\boldsymbol{A}_d+\Delta\boldsymbol{A}_d)\chi(u-d(u))$$

$$=\left[\gamma_1(u)\ \ \gamma_2(u)\cdots\gamma_n(u)\right]^T, \tag{1b}$$

$$\chi(u)=\varphi(u),\forall u\in[-h_2,0] \tag{1c}$$

where $\chi(u)\in\mathbb{R}^n$ is the state vector; $\boldsymbol{A},\ \boldsymbol{A}_d\in\mathbb{R}^{n\times n}$ are the known constant matrices; The unknown matrices $\Delta\boldsymbol{A},\ \Delta\boldsymbol{A}_d\in\mathbb{R}^{n\times n}$ denoting parametric uncertainties in the state matrices; $\boldsymbol{Q}(\cdot)$ and $\boldsymbol{O}(\cdot)$ signify quantization and overflow nonlinearities, respectively; $\boldsymbol{\lambda}(\cdot)$ denotes composite nonlinear effects including both quantization and overflow. Here, $\boldsymbol{Q}(\cdot)$ being either roundoff or magnitude truncation and $\boldsymbol{\lambda}(\cdot)$ is confined to the sector $[k_o, k_q]$, that is,

$$\lambda_i(0)=0,\ \ k_o\gamma_i^2(u)\le\lambda_i(\gamma_i(u))\gamma_i(u)\le k_q\gamma_i^2(u),\ i=1,2,\ldots,n \tag{2a}$$

where

$$k_q=\begin{cases}1, & \text{for magnitude truncation,}\\ 2, & \text{for roundoff,}\end{cases}$$

$$k_o=\begin{cases}0, & \text{for saturation or zeroing,}\\ -\dfrac{1}{3}, & \text{for triangular,}\\ -1, & \text{for 2's complement,}\end{cases} \tag{2b}$$

and $d(u)$ is the variable time-lag that satisfy

$$1\le h_1\le d(u)\le h_2 \tag{3}$$

where h_1 and h_2 are the lower and upper admissible time-lag limits, respectively. $\Delta\boldsymbol{A},\ \Delta\boldsymbol{A}_d\in\mathbb{R}^{n\times n}$ are considered as

$$\Delta\boldsymbol{A}=\boldsymbol{H}_0\boldsymbol{F}_0\boldsymbol{E}_0, \tag{4a}$$

$$\Delta\boldsymbol{A}_d=\boldsymbol{H}_1\boldsymbol{F}_1\boldsymbol{E}_1 \tag{4b}$$

where $\boldsymbol{H}_i\in\mathbb{R}^{n\times p_i}$, $\boldsymbol{E}_i\in\mathbb{R}^{q_i\times n}$ (i = 0, 1) are constant matrices (known) and $\boldsymbol{F}_i\in\mathbb{R}^{p_i\times q_i}$ (i = 0, 1) is a matrix (unknown) which fulfills

$$\boldsymbol{F}_i^T\boldsymbol{F}_i\le\boldsymbol{I},\ \ i=0,1. \tag{4c}$$

Lemma 1 [26, 28] For a matrix $\boldsymbol{R}=\boldsymbol{R}^T>\mathbf{0}$, any matrices $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ of appropriate dimensions, integers a_1 and a_2 satisfying $a_1<a_2$ such that (5) holds

$$-\sum_{s=u-a_2}^{u-a_1-1}\boldsymbol{\eta}^T(s)\boldsymbol{R}\boldsymbol{\eta}(s)\le(a_2-a_1)\begin{bmatrix}\boldsymbol{k}_1\\ \boldsymbol{k}_2\end{bmatrix}^T\begin{bmatrix}\boldsymbol{\Omega}_1 & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\Omega}_2\end{bmatrix}\begin{bmatrix}\boldsymbol{k}_1\\ \boldsymbol{k}_2\end{bmatrix}$$

$$-2\begin{bmatrix}\boldsymbol{k}_1\\ \boldsymbol{k}_2\end{bmatrix}^T\begin{bmatrix}\boldsymbol{L}_1 & \mathbf{0}\\ \mathbf{0} & \boldsymbol{L}_2\end{bmatrix}\begin{bmatrix}\boldsymbol{k}_1\\ \boldsymbol{k}_2\end{bmatrix} \tag{5}$$

where

$$\Gamma(s)=\chi(s+1)-\chi(s),$$

$$\boldsymbol{k}_1 = \chi(u-a_1) - \chi(u-a_2),$$

$$\boldsymbol{k}_2 = \chi(u-a_1) + \chi(u-a_2) - \sum_{i=u-a_2}^{u-a_1} \frac{2\chi(i)}{a_2 - a_1 + 1},$$

$\boldsymbol{\Omega}_1 = \boldsymbol{L}_1\boldsymbol{R}^{-1}\boldsymbol{L}_1^T$ and $\boldsymbol{\Omega}_2 = \boldsymbol{L}_2(3\boldsymbol{R})^{-1}\boldsymbol{L}_2^T$.

Lemma 2 [16, 37, 38] Let $\boldsymbol{\Sigma}$, Λ, $\boldsymbol{F}$ and $\boldsymbol{M}$ be real matrices of appropriate sizes with $\boldsymbol{M}$ fulfilling $\boldsymbol{M} = \boldsymbol{M}^T$, then

$$\boldsymbol{M} + \boldsymbol{\Sigma}\boldsymbol{F}\Lambda + \Lambda^T\boldsymbol{F}^T\boldsymbol{\Sigma}^T < \mathbf{0} \tag{6}$$

for all $\boldsymbol{F}^T\boldsymbol{F} \le \boldsymbol{I}$, if and only if there exists a scalar $\varepsilon > 0$ satisfying

$$\boldsymbol{M} + \varepsilon^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T + \varepsilon\Lambda^T\Lambda < \mathbf{0}. \tag{7}$$

Remark 1 In [26], Lemma 1 has been used for stability analysis of discrete-time systems having variable time-lag but not having quantization/overflow nonlinearities and parameter uncertainties.

Now, we are presenting the stability conditions of the paper.

3 Main Result

Theorem 1. For known integers h_1 and h_2, the system (1)-(4) is globally asymptotically stable (GAS) if there exist matrices $\mathbf{0} < \boldsymbol{P} = \begin{bmatrix} \boldsymbol{P}_1 & \boldsymbol{P}_2 & \boldsymbol{P}_3 & \boldsymbol{P}_4 \\ * & \boldsymbol{P}_5 & \boldsymbol{P}_6 & \boldsymbol{P}_7 \\ * & * & \boldsymbol{P}_8 & \boldsymbol{P}_9 \\ * & * & * & \boldsymbol{P}_{10} \end{bmatrix} = \boldsymbol{P}^T \in \mathbb{R}^{4n \times 4n}$, $\mathbf{0} < \boldsymbol{M}_i = \boldsymbol{M}_i^T (i = 1,2,3) \in \mathbb{R}^{n\times n}$, $\mathbf{0} < \boldsymbol{N}_i = \boldsymbol{N}_i^T (i = 1,2) \in \mathbb{R}^{n\times n}$, any matrices $\boldsymbol{L}_i$ $(i = 1,2,3,...,6) \in \mathbb{R}^{n\times n}$, $\mathbf{0} < \boldsymbol{G} = \boldsymbol{G}^T = diag(g_1, g_2, \ldots, g_n)$ and two positive scalars $\varepsilon_0, \varepsilon_1$ such that LMIs (8a) and (8b) hold simultaneously:

$$\Psi(d(u) = h_1) < \mathbf{0}, \tag{8a}$$

$$\Psi(d(u) = h_2) < \mathbf{0} \tag{8b}$$

where

$$\Psi(d(u)) =$$

$$\left[\begin{array}{cccccc}
\boldsymbol{Q}_{11} + \varepsilon_0\boldsymbol{E}_0^T\boldsymbol{E}_0 & \boldsymbol{Q}_{12} & -\boldsymbol{P}_7 + \boldsymbol{P}_4 & 0 & \boldsymbol{Q}_{15} & \boldsymbol{Q}_{16} \\
* & \boldsymbol{Q}_{22} & \boldsymbol{Q}_{23} & \boldsymbol{P}_7 & \boldsymbol{Q}_{25} & \boldsymbol{Q}_{26} \\
* & * & \boldsymbol{Q}_{33} + \varepsilon_1\boldsymbol{E}_1^T\boldsymbol{E}_1 & \boldsymbol{Q}_{34} & \boldsymbol{Q}_{35} & \boldsymbol{Q}_{36} \\
* & * & * & \boldsymbol{Q}_{44} & -\boldsymbol{P}_7^T(h_1+1) & \boldsymbol{Q}_{46} \\
* & * & * & * & -4h_1(\boldsymbol{L}_2 + \boldsymbol{L}_2^T) & \mathbf{0} \\
* & * & * & * & * & \boldsymbol{Q}_{66} \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\
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* & * & * & * & * & * \\
* & * & * & * & * & *
\end{array}\right.$$

$$\begin{array}{ccccc}
\boldsymbol{Q}_{17} & \boldsymbol{Q}_{18} & h_1\boldsymbol{L}_1 & h_1\boldsymbol{L}_2 & \mathbf{0} \\
\boldsymbol{Q}_{27} & -\boldsymbol{P}_2^T & -h_1\boldsymbol{L}_1 & h_1\boldsymbol{L}_2 & Q_{211}\boldsymbol{L}_3 \\
\boldsymbol{Q}_{37} & -\boldsymbol{P}_3^T + k_q\boldsymbol{A}_d^T\boldsymbol{G} & \mathbf{0} & \mathbf{0} & -Q_{211}\boldsymbol{L}_3 \\
\boldsymbol{Q}_{47} & -\boldsymbol{P}_4^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \boldsymbol{P}_2^T(h_1+1) & \mathbf{0} & -2h_1\boldsymbol{L}_2 & \mathbf{0} \\
\mathbf{0} & \boldsymbol{Q}_{68} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\boldsymbol{Q}_{77} & \boldsymbol{Q}_{78} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & \boldsymbol{Q}_{88} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & -\boldsymbol{N}_1 & \mathbf{0} & \mathbf{0} \\
* & * & * & -3\boldsymbol{N}_1 & \mathbf{0} \\
* & * & * & * & -\boldsymbol{N}_2 \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & *
\end{array}$$

$$\left.\begin{array}{cccccc}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{Q}_{115} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & Q_{211}\boldsymbol{L}_4 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
Q_{312}\boldsymbol{L}_5 & Q_{211}\boldsymbol{L}_4 & Q_{312}\boldsymbol{L}_6 & \boldsymbol{Q}_{315} & \mathbf{0} & \mathbf{0} \\
-Q_{312}\boldsymbol{L}_5 & \mathbf{0} & Q_{312}\boldsymbol{L}_6 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & -2Q_{211}\boldsymbol{L}_4 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & -2Q_{312}\boldsymbol{L}_6 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \sqrt{-k_o/2}\,\boldsymbol{G} & k_q\boldsymbol{G}\boldsymbol{H}_0 & k_q\boldsymbol{G}\boldsymbol{H}_1 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
-\boldsymbol{N}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & -3\boldsymbol{N}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & -3\boldsymbol{N}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & -k_q\boldsymbol{G} & -k_q\sqrt{-2k_o}\boldsymbol{G}\boldsymbol{H}_0 & -k_q\sqrt{-2k_o}\boldsymbol{G}\boldsymbol{H}_1 \\
* & * & * & * & -\varepsilon_0\boldsymbol{I} & \mathbf{0} \\
* & * & * & * & * & -\varepsilon_1\boldsymbol{I}
\end{array}\right], \tag{9}$$

$$\boldsymbol{Q}_{11}=-\boldsymbol{P}_5-\boldsymbol{P}_1+\boldsymbol{P}_2^T+\boldsymbol{P}_2+\boldsymbol{M}_1+h_1^2\boldsymbol{N}_1+h_{12}^2\boldsymbol{N}_2 -h_1(\boldsymbol{L}_1^T+\boldsymbol{L}_1+\boldsymbol{L}_2^T+\boldsymbol{L}_2)+(h_{12}+1)\boldsymbol{M}_3, \quad (10)$$

$$\boldsymbol{Q}_{12}=\boldsymbol{P}_3-\boldsymbol{P}_6+h_1(\boldsymbol{L}_1^T+\boldsymbol{L}_1-\boldsymbol{L}_2^T-\boldsymbol{L}_2), \quad (11)$$

$$\boldsymbol{Q}_{15}=(h_1+1)(\boldsymbol{P}_5-\boldsymbol{P}_2)+2h_1(\boldsymbol{L}_2^T+\boldsymbol{L}_2), \quad (12)$$

$$\boldsymbol{Q}_{16}=(d(u)-h_1+1)(\boldsymbol{P}_6-\boldsymbol{P}_3), \quad (13)$$

$$\boldsymbol{Q}_{17}=(h_2-d(u)+1)(\boldsymbol{P}_7-\boldsymbol{P}_4), \quad (14)$$

$$\boldsymbol{Q}_{18}=-(h_1^2\boldsymbol{N}_1+h_{12}^2\boldsymbol{N}_2)+k_q\boldsymbol{A}^T\boldsymbol{G}, \quad (15)$$

$$\boldsymbol{Q}_{115}=-k_q\sqrt{-2k_o}\boldsymbol{A}^T\boldsymbol{G}, \quad (16)$$

$$\boldsymbol{Q}_{22}=\boldsymbol{P}_5-\boldsymbol{P}_8+\boldsymbol{M}_2-\boldsymbol{M}_1-h_1(\boldsymbol{L}_1+\boldsymbol{L}_2+\boldsymbol{L}_1^T+\boldsymbol{L}_2^T) -h_{12}(\boldsymbol{L}_3+\boldsymbol{L}_4+\boldsymbol{L}_3^T+\boldsymbol{L}_4^T), \quad (17)$$

$$\boldsymbol{Q}_{23}=\boldsymbol{P}_6-\boldsymbol{P}_9+h_{12}(\boldsymbol{L}_3^T+\boldsymbol{L}_3-\boldsymbol{L}_4^T-\boldsymbol{L}_4), \quad (18)$$

$$\boldsymbol{Q}_{25}=(-\boldsymbol{P}_5+\boldsymbol{P}_6^T)(h_1+1)+2h_1(\boldsymbol{L}_2^T+\boldsymbol{L}_2), \quad (19)$$

$$\boldsymbol{Q}_{26}=(d(u)-h_1+1)(\boldsymbol{P}_8-\boldsymbol{P}_6)+2h_{12}(\boldsymbol{L}_4^T+\boldsymbol{L}_4), \quad (20)$$

$$\boldsymbol{Q}_{27}=(\boldsymbol{P}_9-\boldsymbol{P}_7)(h_2-d(u)+1), \quad (21)$$

$$Q_{211}=\sqrt{h_{12}}\sqrt{d(u)-h_1}, \quad (22)$$

$$Q_{312}=\sqrt{h_{12}}\sqrt{h_2-d(u)}, \quad (23)$$

$$\boldsymbol{Q}_{315}=-k_q\sqrt{-2k_o}\boldsymbol{A}_d^T\boldsymbol{G}, \quad (24)$$

$$\boldsymbol{Q}_{33}=\boldsymbol{P}_8-\boldsymbol{P}_{10}-\boldsymbol{M}_3 -h_{12}(\boldsymbol{L}_3+\boldsymbol{L}_4+\boldsymbol{L}_5+\boldsymbol{L}_6+\boldsymbol{L}_3^T+\boldsymbol{L}_4^T+\boldsymbol{L}_5^T+\boldsymbol{L}_6^T), \quad (25)$$

$$\boldsymbol{Q}_{34}=\boldsymbol{P}_9+h_{12}(\boldsymbol{L}_5^T+\boldsymbol{L}_5-\boldsymbol{L}_6^T-\boldsymbol{L}_6), \quad (26)$$

$$\boldsymbol{Q}_{35}=(h_1+1)(\boldsymbol{P}_7^T-\boldsymbol{P}_6^T), \quad (27)$$

$$\boldsymbol{Q}_{36}=(d(u)-h_1+1)(\boldsymbol{P}_9^T-\boldsymbol{P}_8^T)+2h_{12}(\boldsymbol{L}_4^T+\boldsymbol{L}_4), \quad (28)$$

$$\boldsymbol{Q}_{37}=(h_2-d(u)+1)(\boldsymbol{P}_{10}-\boldsymbol{P}_9)+2h_{12}(\boldsymbol{L}_6^T+\boldsymbol{L}_6), \quad (29)$$

$$\boldsymbol{Q}_{44}=\boldsymbol{P}_{10}-\boldsymbol{M}_2-h_{12}(\boldsymbol{L}_5^T+\boldsymbol{L}_5+\boldsymbol{L}_6^T+\boldsymbol{L}_6), \quad (30)$$

$$\boldsymbol{Q}_{46}=-\boldsymbol{P}_9^T(d(u)-h_1+1), \quad (31)$$

$$\boldsymbol{Q}_{47}=-\boldsymbol{P}_{10}(h_2-d(u)+1)+2h_{12}(\boldsymbol{L}_6^T+\boldsymbol{L}_6), \quad (32)$$

$$\boldsymbol{Q}_{66}=-4h_{12}(\boldsymbol{L}_4+\boldsymbol{L}_4^T), \quad (33)$$

$$\boldsymbol{Q}_{68}=\boldsymbol{P}_3^T(d(u)-h_1+1), \quad (34)$$

$$\boldsymbol{Q}_{77}=-4h_{12}(\boldsymbol{L}_6+\boldsymbol{L}_6^T), \quad (35)$$

$$\boldsymbol{Q}_{78}=\boldsymbol{P}_4^T(h_2-d(u)+1), \quad (36)$$

$$h_{12}=h_2-h_1, \quad (37)$$

$$\boldsymbol{Q}_{88}=\boldsymbol{P}_1+h_1^2\boldsymbol{N}_1+h_{12}^2\boldsymbol{N}_2+[(k_0/2k_q)-2]\boldsymbol{G}. \quad (38)$$

Proof. Consider a quadratic Lyapunov function

$$V(\chi(u))=\eta^T(u)\boldsymbol{P}\eta(u)+\sum_{s=u-h_1}^{u-1}\chi^T(s)\boldsymbol{M}_1\chi(s)+\sum_{s=u-h_2}^{u-h_1-1}\chi^T(s)\boldsymbol{M}_2\chi(s) +\sum_{\theta=-h_2}^{-h_1}\sum_{s=u+\theta}^{u-1}\chi^T(s)\boldsymbol{M}_3\chi(s)+h_1\sum_{\theta=-h_1}^{-1}\sum_{s=u+\theta}^{u-1}\Gamma^T(s)\boldsymbol{N}_1\Gamma(s) +h_{12}\sum_{\theta=-h_2}^{-h_1-1}\sum_{s=u+\theta}^{u-1}\Gamma^T(s)\boldsymbol{N}_2\Gamma(s) \quad (39)$$

where

$$\eta^T(u)=[\chi^T(u)\ \ \sum_{s=u-h_1}^{u-1}\chi^T(s)\ \ \sum_{s=u-d(u)}^{u-h_1-1}\chi^T(s)\ \ \sum_{s=u-h_2}^{u-d(u)-1}\chi^T(s)] \quad (40)$$

and

$$\Gamma(u)=\chi(u+1)-\chi(u)=\boldsymbol{\lambda}(\gamma(u))-\chi(u). \quad (41)$$

Application of (39) to (1) yields

$$\begin{aligned}\Delta V(\chi(u))&=V(\chi(u+1))-V(\chi(u))\\ &=\eta^T(u+1)\boldsymbol{P}\eta(u+1)-\eta^T(u)\boldsymbol{P}\eta(u)\\ &\quad+\chi^T(u)\boldsymbol{M}_1\chi(u)-\chi^T(u-h_1)(\boldsymbol{M}_1-\boldsymbol{M}_2)\chi(u-h_1)\\ &\quad-\chi^T(u-h_2)\boldsymbol{M}_2\chi(u-h_2)\\ &\quad+\chi^T(u)\boldsymbol{M}_3\chi(u)+h_{12}\chi^T(u)\boldsymbol{M}_3\chi(u)\\ &\quad-\sum_{s=u-h_2}^{u-h_1}\chi^T(s)\boldsymbol{M}_3\chi(s)\\ &\quad+\Gamma^T(u)[h_1^2\boldsymbol{N}_1+h_{12}^2\boldsymbol{N}_2]\Gamma(u)\\ &\quad-h_1\sum_{s=u-h_1}^{u-1}\Gamma^T(s)\boldsymbol{N}_1\Gamma(s)\\ &\quad-h_{12}\sum_{s=u-h_2}^{u-h_1-1}\Gamma^T(s)\boldsymbol{N}_2\Gamma(s). \end{aligned} \quad (42)$$

Now, (42) can be further rearranged as

$$\begin{aligned}\Delta V(\chi(u))&=\Xi^T(u)\Theta(d(u))\Xi(u)+\chi^T(u)\boldsymbol{M}_1\chi(u)\\ &\quad-\chi^T(u-h_1)(\boldsymbol{M}_1-\boldsymbol{M}_2)\chi(u-h_1)\\ &\quad-\chi^T(u-h_2)\boldsymbol{M}_2\chi(u-h_2)+\chi^T(u)\boldsymbol{M}_3\chi(u)\\ &\quad+h_{12}\chi^T(u)\boldsymbol{M}_3\chi(u)-\sum_{s=u-h_2}^{u-h_1}\chi^T(s)\boldsymbol{M}_3\chi(s)\\ &\quad+\Gamma^T(u)[h_1^2\boldsymbol{N}_1+h_{12}^2\boldsymbol{N}_2]\Gamma(u)\\ &\quad-h_1\sum_{s=u-h_1}^{u-1}\Gamma^T(s)\boldsymbol{N}_1\Gamma(s)-h_{12}\sum_{s=u-h_2}^{u-h_1-1}\Gamma^T(s)\boldsymbol{N}_2\Gamma(s)\end{aligned} \quad (43)$$

where

$$\Xi^T(u)=\Big[\chi^T(u)\ \chi^T(u-h_1)\ \chi^T(u-d(u))\ \ \chi^T(u-h_2)\ \ \frac{1}{h_1+1}\sum_{s=u-h_1}^{u}\chi^T(s)\ \ \frac{1}{d(u)-h_1+1}\sum_{s=u-d(u)}^{u-h_1}\chi^T(s)\ \ \frac{1}{h_2-d(u)+1}\sum_{s=u-h_2}^{u-d(u)}\chi^T(s)\ \ \boldsymbol{\lambda}^T(\gamma(u))\Big], \quad (44)$$

$$\boldsymbol{\Theta}(d(u)) = \begin{bmatrix} -\boldsymbol{P}_5-\boldsymbol{P}_1+\boldsymbol{P}_2^T+\boldsymbol{P}_2 & \boldsymbol{P}_3-\boldsymbol{P}_6 & -\boldsymbol{P}_7+\boldsymbol{P}_4 & \mathbf{0} & (\boldsymbol{P}_5-\boldsymbol{P}_2)(h_1+1) & \boldsymbol{Q}_{16} & \boldsymbol{Q}_{17} & \mathbf{0} \\ * & \boldsymbol{P}_5-\boldsymbol{P}_8 & \boldsymbol{P}_6-\boldsymbol{P}_9 & \boldsymbol{P}_7 & (-\boldsymbol{P}_5+\boldsymbol{P}_6^T)(h_1+1) & (\boldsymbol{P}_8-\boldsymbol{P}_6)(d(u)-h_1+1) & \boldsymbol{Q}_{27} & -\boldsymbol{P}_2^T \\ * & * & \boldsymbol{P}_8-\boldsymbol{P}_{10} & \boldsymbol{P}_9 & (\boldsymbol{P}_7^T-\boldsymbol{P}_6^T)(h_1+1) & (\boldsymbol{P}_9^T-\boldsymbol{P}_8^T)(d(u)-h_1+1) & (\boldsymbol{P}_{10}-\boldsymbol{P}_9)(h_2-d(u)+1) & -\boldsymbol{P}_3^T \\ * & * & * & \boldsymbol{P}_{10} & (-\boldsymbol{P}_7^T)(h_1+1) & \boldsymbol{Q}_{46} & -\boldsymbol{P}_{10}(h_2-d(u)+1) & -\boldsymbol{P}_4^T \\ * & * & * & * & \mathbf{0} & \mathbf{0} & \mathbf{0} & (h_1+1)\boldsymbol{P}_2^T \\ * & * & * & * & * & \mathbf{0} & \mathbf{0} & \boldsymbol{Q}_{68} \\ * & * & * & * & * & * & \mathbf{0} & \boldsymbol{Q}_{78} \\ * & * & * & * & * & * & * & \boldsymbol{P}_1 \end{bmatrix}. \quad (45)$$

Note that

$$-\sum_{s=u-h_2}^{u-h_1} \chi^T(s)\boldsymbol{M}_3\chi(s) \le -\chi^T(u-d(u))\boldsymbol{M}_3\chi(u-d(u)). \quad (46)$$

Next, applying Lemma 1 to deal the 9th and 10th terms in the right side of (43), we have

$$-h_1\sum_{s=u-h_1}^{u-1}\Gamma^T(s)\boldsymbol{N}_1\Gamma(s)$$

$$\le h_1^2\begin{bmatrix}\boldsymbol{k}_1\\ \boldsymbol{k}_2\end{bmatrix}^T\begin{bmatrix}\boldsymbol{L}_1\boldsymbol{N}_1^{-1}\boldsymbol{L}_1^{\mathrm{T}} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{L}_2(3\boldsymbol{N}_1)^{-1}\boldsymbol{L}_2^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\boldsymbol{k}_1\\ \boldsymbol{k}_2\end{bmatrix}$$

$$-2h_1\begin{bmatrix}\boldsymbol{k}_1\\ \boldsymbol{k}_2\end{bmatrix}^T\begin{bmatrix}\boldsymbol{L}_1 & \mathbf{0}\\ \mathbf{0} & \boldsymbol{L}_2\end{bmatrix}\begin{bmatrix}\boldsymbol{k}_1\\ \boldsymbol{k}_2\end{bmatrix}$$

$$= h_1^2\begin{bmatrix}\boldsymbol{k}_1\\ \boldsymbol{k}_2\end{bmatrix}^T\begin{bmatrix}\boldsymbol{L}_1\boldsymbol{N}_1^{-1}\boldsymbol{L}_1^{\mathrm{T}} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{L}_2(3\boldsymbol{N}_1)^{-1}\boldsymbol{L}_2^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\boldsymbol{k}_1\\ \boldsymbol{k}_2\end{bmatrix}$$

$$-h_1\begin{bmatrix}\boldsymbol{k}_1\\ \boldsymbol{k}_2\end{bmatrix}^T\begin{bmatrix}\boldsymbol{L}_1+\boldsymbol{L}_1^{\mathrm{T}} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{L}_2+\boldsymbol{L}_2^T\end{bmatrix}\begin{bmatrix}\boldsymbol{k}_1\\ \boldsymbol{k}_2\end{bmatrix} \quad (47)$$

and

$$-\sum_{s=u-h_2}^{u-h_1-1}\Gamma^T(s)\boldsymbol{N}_2\Gamma(s)$$

$$= -\sum_{s=u-d(u)}^{u-h_1-1}\Gamma^T(s)\boldsymbol{N}_2\Gamma(s) - \sum_{s=u-h_2}^{u-d(u)-1}\Gamma^T(s)\boldsymbol{N}_2\Gamma(s)$$

$$\le (d(u)-h_1)\begin{bmatrix}\boldsymbol{k}_3\\ \boldsymbol{k}_4\end{bmatrix}^T\begin{bmatrix}\boldsymbol{L}_3\boldsymbol{N}_2^{-1}\boldsymbol{L}_3^{\mathrm{T}} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{L}_4(3\boldsymbol{N}_2)^{-1}\boldsymbol{L}_4^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\boldsymbol{k}_3\\ \boldsymbol{k}_4\end{bmatrix}$$

$$-\begin{bmatrix}\boldsymbol{k}_3\\ \boldsymbol{k}_4\end{bmatrix}^T\begin{bmatrix}\boldsymbol{L}_3+\boldsymbol{L}_3^{\mathrm{T}} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{L}_4+\boldsymbol{L}_4^T\end{bmatrix}\begin{bmatrix}\boldsymbol{k}_3\\ \boldsymbol{k}_4\end{bmatrix}$$

$$+(h_2-d(u))\begin{bmatrix}\boldsymbol{k}_5\\ \boldsymbol{k}_6\end{bmatrix}^T\begin{bmatrix}\boldsymbol{L}_5\boldsymbol{N}_2^{-1}\boldsymbol{L}_5^{\mathrm{T}} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{L}_6(3\boldsymbol{N}_2)^{-1}\boldsymbol{L}_6^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\boldsymbol{k}_5\\ \boldsymbol{k}_6\end{bmatrix}$$

$$-\begin{bmatrix}\boldsymbol{k}_5\\ \boldsymbol{k}_6\end{bmatrix}^T\begin{bmatrix}\boldsymbol{L}_5+\boldsymbol{L}_5^{\mathrm{T}} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{L}_6+\boldsymbol{L}_6^T\end{bmatrix}\begin{bmatrix}\boldsymbol{k}_5\\ \boldsymbol{k}_6\end{bmatrix} \quad (48)$$

where

$$\boldsymbol{k}_1 = \chi(u)-\chi(u-h_1), \quad (49a)$$

$$\boldsymbol{k}_2 = \chi(u)+\chi(u-h_1)-\sum_{i=u-h_1}^{u}\frac{2\chi(i)}{h_1+1}, \quad (49b)$$

$$\boldsymbol{k}_3 = \chi(u-h_1)-\chi(u-d(u)), \quad (49c)$$

$$\boldsymbol{k}_4 = \chi(u-h_1)+\chi(u-d(u))-\sum_{i=u-d(u)}^{u-h_1}\frac{2\chi(i)}{d(u)-h_1+1}, \quad (49d)$$

$$\boldsymbol{k}_5 = \chi(u-d(u))-\chi(u-h_2), \quad (49e)$$

and

$$\boldsymbol{k}_6 = \chi(u-d(u))+\chi(u-h_2)-\sum_{i=u-h_2}^{u-d(u)}\frac{2\chi(i)}{h_2-d(u)+1}. \quad (49f)$$

Employing (43)-(49), we have

$$\Delta V(\chi(u)) = \Xi^T(u)\boldsymbol{\Pi}(d(u))\Xi(u)-2\beta, \quad (50)$$

where

$$\beta = \sum_{i=1}^{n} g_i[k_q\gamma_i(u)-\lambda_i(\gamma_i(u))][\lambda_i(\gamma_i(u))-k_o\gamma_i(u)]$$

$$= [k_q\gamma(u)-\boldsymbol{\lambda}(\gamma(u))]^T\boldsymbol{G}[\boldsymbol{\lambda}(\gamma(u)-k_o\gamma(u)], \quad (51)$$

$$\boldsymbol{\Pi}(d(u)) = \left[\begin{matrix} \boldsymbol{Q}_{11}-2k_ok_q\bar{\boldsymbol{A}}^T\boldsymbol{G}\bar{\boldsymbol{A}} + h_1^2(\boldsymbol{L}_1\boldsymbol{N}_1^{-1}\boldsymbol{L}_1^T)+h_1^2(\boldsymbol{L}_2(3\boldsymbol{N}_1)^{-1}\boldsymbol{L}_2^T) & \boldsymbol{Q}_{12}-h_1^2(\boldsymbol{L}_1\boldsymbol{N}_1^{-1}\boldsymbol{L}_1^T)+h_1^2(\boldsymbol{L}_2(3\boldsymbol{N}_1)^{-1}\boldsymbol{L}_2^T) \\ * & \boldsymbol{Q}_{22}+h_1^2(\boldsymbol{L}_1\boldsymbol{N}_1^{-1}\boldsymbol{L}_1^T)+h_1^2(\boldsymbol{L}_2(3\boldsymbol{N}_1)^{-1}\boldsymbol{L}_2^T) + h_{12}(d(u)-h_1)(\boldsymbol{L}_3\boldsymbol{N}_2^{-1}\boldsymbol{L}_3^T)+h_{12}(d(u)-h_1)(\boldsymbol{L}_4(3\boldsymbol{N}_2)^{-1}\boldsymbol{L}_4^T) \\ * & * \\ * & * \\ * & * \\ * & * \\ * & * \\ * & * \end{matrix}\right.$$

$$\begin{array}{cc}
-\boldsymbol{P}_7+\boldsymbol{P}_4-2k_ok_q\bar{\boldsymbol{A}}^T\boldsymbol{G}\bar{\boldsymbol{A}}_d & \mathbf{0} \\
\boldsymbol{Q}_{23}-h_{12}(d(u)-h_1)(\boldsymbol{L}_3\boldsymbol{N}_2^{-1}\boldsymbol{L}_3^T)+h_{12}(d(u)-h_1)(\boldsymbol{L}_4(3\boldsymbol{N}_2)^{-1}\boldsymbol{L}_4^T) & \boldsymbol{P}_7 \\
\boldsymbol{Q}_{33}-2k_ok_q\bar{\boldsymbol{A}}_d^T\boldsymbol{G}\bar{\boldsymbol{A}}_d+h_{12}(d(u)-h_1)(\boldsymbol{L}_3\boldsymbol{N}_2^{-1}\boldsymbol{L}_3^T)+h_{12}(d(u)-h_1)(\boldsymbol{L}_4(3\boldsymbol{N}_2)^{-1}\boldsymbol{L}_4^T)+\boldsymbol{Z}_{11}+\boldsymbol{Z}_{12} & \boldsymbol{Q}_{34}-\boldsymbol{Z}_{11}+\boldsymbol{Z}_{12} \\
* & \boldsymbol{Q}_{44}+\boldsymbol{Z}_{11}+\boldsymbol{Z}_{12} \\
* & * \\
* & * \\
* & * \\
* & *
\end{array}$$

$$\left.\begin{array}{cccc}
\boldsymbol{Q}_{15}-2h_1^2(\boldsymbol{L}_2(3\boldsymbol{N}_1)^{-1}\boldsymbol{L}_2^T) & \boldsymbol{Q}_{16} & \boldsymbol{Q}_{17} & -(h_1^2\boldsymbol{N}_1+h_{12}^2\boldsymbol{N}_2)+(k_q+k_o)\bar{\boldsymbol{A}}^T\boldsymbol{G} \\
\boldsymbol{Q}_{25}-2h_1^2(\boldsymbol{L}_2(3\boldsymbol{N}_1)^{-1}\boldsymbol{L}_2^T) & \boldsymbol{Q}_{26}-2h_{12}(d(u)-h_1)(\boldsymbol{L}_4(3\boldsymbol{N}_2)^{-1}\boldsymbol{L}_4^T) & \boldsymbol{Q}_{27} & -\boldsymbol{P}_2^T \\
\boldsymbol{Q}_{35} & \boldsymbol{Q}_{36}-2h_{12}(d(u)-h_1)(\boldsymbol{L}_4(3\boldsymbol{N}_2)^{-1}\boldsymbol{L}_4^T) & \boldsymbol{Q}_{37}-2\boldsymbol{Z}_{12} & -\boldsymbol{P}_3^T+(k_q+k_o)\bar{\boldsymbol{A}}_d^T\boldsymbol{G} \\
-\boldsymbol{P}_7^T(h_1+1) & \boldsymbol{Q}_{46} & \boldsymbol{Q}_{47}-2\boldsymbol{Z}_{12} & -\boldsymbol{P}_4^T \\
-4h_1(\boldsymbol{L}_2+\boldsymbol{L}_2^T)+4h_1^2(\boldsymbol{L}_2(3\boldsymbol{N}_1)^{-1}\boldsymbol{L}_2^T) & \mathbf{0} & \mathbf{0} & \boldsymbol{P}_2^T(h_1+1) \\
* & \boldsymbol{Q}_{66}+4h_{12}(d(u)-h_1)(\boldsymbol{L}_4(3\boldsymbol{N}_2)^{-1}\boldsymbol{L}_4^T) & \mathbf{0} & \boldsymbol{Q}_{68} \\
* & * & \boldsymbol{Q}_{77}+4\boldsymbol{Z}_{12} & \boldsymbol{Q}_{78} \\
* & * & * & \boldsymbol{P}_1+h_1^2\boldsymbol{N}_1+h_{12}^2\boldsymbol{N}_2-2\boldsymbol{G}
\end{array}\right], \quad (52)$$

$$\boldsymbol{Z}_{11}=h_{12}(h_2-d(u))(\boldsymbol{L}_5\boldsymbol{N}_2^{-1}\boldsymbol{L}_5^T), \quad (53a)$$

$$\boldsymbol{Z}_{12}=h_{12}(h_2-d(u))(\boldsymbol{L}_6(3\boldsymbol{N}_2)^{-1}\boldsymbol{L}_6^T), \quad (53b)$$

and $\bar{\boldsymbol{A}}=\boldsymbol{A}+\Delta\boldsymbol{A}$, $\bar{\boldsymbol{A}}_d=\boldsymbol{A}_d+\Delta\boldsymbol{A}_d$. (54)

In view of (2), (51) is non-negative. From (50) and (51), it follows that $\Delta V(\chi(u))<0$ if $\boldsymbol{\Pi}(d(u))<\mathbf{0}$ for $d(u)\in[h_1,h_2]$. Therefore, $\boldsymbol{\Pi}(d(u))<\mathbf{0}$ is a sufficient condition for the stability of the system (1)-(4).

By exploiting the concept of Schur's complement, $\boldsymbol{\Pi}(d(u))<\mathbf{0}$ is equivalent to

$$\left[\begin{array}{cccccc}
\boldsymbol{Q}_{11}-2k_ok_q\bar{\boldsymbol{A}}^T\boldsymbol{G}\bar{\boldsymbol{A}} & \boldsymbol{Q}_{12} & -\boldsymbol{P}_7+\boldsymbol{P}_4-2k_ok_q\bar{\boldsymbol{A}}^T\boldsymbol{G}\bar{\boldsymbol{A}}_d & \mathbf{0} & \boldsymbol{Q}_{15} & \boldsymbol{Q}_{16} \\
* & \boldsymbol{Q}_{22} & \boldsymbol{Q}_{23} & \boldsymbol{P}_7 & \boldsymbol{Q}_{25} & \boldsymbol{Q}_{26} \\
* & * & \boldsymbol{Q}_{33}-2k_ok_q\bar{\boldsymbol{A}}_d^T\boldsymbol{G}\bar{\boldsymbol{A}}_d & \boldsymbol{Q}_{34} & \boldsymbol{Q}_{35} & \boldsymbol{Q}_{36} \\
* & * & * & \boldsymbol{Q}_{44} & -\boldsymbol{P}_7^T(h_1+1) & \boldsymbol{Q}_{46} \\
* & * & * & * & -4h_1(\boldsymbol{L}_2+\boldsymbol{L}_2^T) & \mathbf{0} \\
* & * & * & * & * & \boldsymbol{Q}_{66} \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & *
\end{array}\right.$$

$$\begin{array}{cccc}
\boldsymbol{Q}_{17} & -(h_1^2\boldsymbol{N}_1+h_{12}^2\boldsymbol{N}_2)+(k_q+k_o)\bar{\boldsymbol{A}}^T\boldsymbol{G} & h_1\boldsymbol{L}_1 & h_1\boldsymbol{L}_2 \\
\boldsymbol{Q}_{27} & -\boldsymbol{P}_2^T & -h_1\boldsymbol{L}_1 & h_1\boldsymbol{L}_2 \\
\boldsymbol{Q}_{37} & -\boldsymbol{P}_3^T+(k_q+k_o)\bar{\boldsymbol{A}}_d^T\boldsymbol{G} & \mathbf{0} & \mathbf{0} \\
\boldsymbol{Q}_{47} & -\boldsymbol{P}_4^T & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \boldsymbol{P}_2^T(h_1+1) & \mathbf{0} & -2h_1\boldsymbol{L}_2 \\
\mathbf{0} & \boldsymbol{Q}_{68} & \mathbf{0} & \mathbf{0} \\
\boldsymbol{Q}_{77} & \boldsymbol{Q}_{78} & \mathbf{0} & \mathbf{0} \\
* & \boldsymbol{P}_1+h_1^2\boldsymbol{N}_1+h_{12}^2\boldsymbol{N}_2-2\boldsymbol{G} & \mathbf{0} & \mathbf{0} \\
* & * & -\boldsymbol{N}_1 & \mathbf{0} \\
* & * & * & -3\boldsymbol{N}_1 \\
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
* & * & * & *
\end{array}$$

$$\left.\begin{array}{cccc}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathcal{Q}_{211}\boldsymbol{L}_3 & \mathbf{0} & \mathcal{Q}_{211}\boldsymbol{L}_4 & \mathbf{0} \\
\mathcal{Q}_{211}\boldsymbol{L}_3 & \mathcal{Q}_{312}\boldsymbol{L}_5 & \mathcal{Q}_{211}\boldsymbol{L}_4 & \mathcal{Q}_{312}\boldsymbol{L}_6 \\
\mathbf{0} & \mathcal{Q}_{312}\boldsymbol{L}_5 & \mathbf{0} & \mathcal{Q}_{312}\boldsymbol{L}_6 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & -2\mathcal{Q}_{211}\boldsymbol{L}_4 & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & -2\mathcal{Q}_{312}\boldsymbol{L}_6 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
-\boldsymbol{N}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & -\boldsymbol{N}_2 & \mathbf{0} & \mathbf{0} \\
* & * & -3\boldsymbol{N}_2 & \mathbf{0} \\
* & * & * & -3\boldsymbol{N}_2
\end{array}\right]<\mathbf{0}, \quad (55)$$

After some mathematical rearrangements and applying Schur's complement, (55) can be rewritten as

$$\begin{bmatrix}
Q_{11} & Q_{12} & -P_7+P_4 & 0 & Q_{15} & Q_{16} & Q_{17} & -(h_1^2N_1+h_{12}^2N_2)+k_q\bar{A}^TG & h_1L_1 & h_1L_2 & 0 & 0 & 0 & 0 & -k_q\sqrt{-2k_o}\bar{A}^TG \\
* & Q_{22} & Q_{23} & P_7 & Q_{25} & Q_{26} & Q_{27} & -P_2^T & -h_1L_1 & h_1L_2 & Q_{211}L_3 & 0 & Q_{211}L_4 & 0 & 0 \\
* & * & Q_{33} & Q_{34} & Q_{35} & Q_{36} & Q_{37} & -P_3^T+k_q\bar{A}_d^TG & 0 & 0 & Q_{211}L_3 & Q_{312}L_5 & Q_{211}L_4 & Q_{312}L_6 & -k_q\sqrt{-2k_o}\bar{A}_d^TG \\
* & * & * & Q_{44} & -P_7^T(h_1+1) & Q_{46} & Q_{47} & -P_4^T & 0 & 0 & 0 & Q_{312}L_5 & 0 & Q_{312}L_6 & 0 \\
* & * & * & * & -4h_1(L_2+L_2^T) & 0 & 0 & P_2^T(h_1+1) & 0 & -2h_1L_2 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & Q_{66} & 0 & Q_{68} & 0 & 0 & 0 & 0 & -2Q_{211}L_4 & 0 & 0 \\
* & * & * & * & * & * & Q_{77} & Q_{78} & 0 & 0 & 0 & 0 & 0 & -2Q_{312}L_6 & 0 \\
* & * & * & * & * & * & * & Q_{88} & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{-k_o/2}G \\
* & * & * & * & * & * & * & * & -N_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -3N_1 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & -N_2 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & -N_2 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & -3N_2 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & -3N_2 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & -k_qG
\end{bmatrix} < 0. \quad (56)$$

Next, with the help of (4a), the inequality (56) can be represented as

$$\boldsymbol{\Pi}_0(d(u)) + \bar{H}_0F_0\bar{E}_0 + \bar{E}_0^TF_0^T\bar{H}_0^T < 0 \quad (57)$$

where

$$\bar{H}_0^T = \Big[\underbrace{0\cdots0}_{7\text{ times}}\ k_qH_0^TG\ \underbrace{0\cdots0}_{6\text{ times}}\ -k_q\sqrt{-2k_0}H_0^TG\Big], \quad (58)$$

$$\bar{E}_0 = [E_0\ \underbrace{0\cdots0}_{14\text{ times}}] \quad (59)$$

and

$$\boldsymbol{\Pi}_0(d(u)) =$$

$$\begin{bmatrix}
Q_{11} & Q_{12} & -P_7+P_4 & 0 & Q_{15} & Q_{16} & Q_{17} & -(h_1^2N_1+h_{12}^2N_2)+k_qA^TG & h_1L_1 & h_1L_2 & 0 & 0 & 0 & 0 & -k_q\sqrt{-2k_o}A^TG \\
* & Q_{22} & Q_{23} & P_7 & Q_{25} & Q_{26} & Q_{27} & -P_2^T & -h_1L_1 & h_1L_2 & Q_{211}L_3 & 0 & Q_{211}L_4 & 0 & 0 \\
* & * & Q_{33} & Q_{34} & Q_{35} & Q_{36} & Q_{37} & -P_3^T+k_q\bar{A}_d^TG & 0 & 0 & Q_{211}L_3 & Q_{312}L_5 & Q_{211}L_4 & Q_{312}L_6 & -k_q\sqrt{-2k_o}\bar{A}_d^TG \\
* & * & * & Q_{44} & -P_7^T(h_1+1) & Q_{46} & Q_{47} & -P_4^T & 0 & 0 & 0 & Q_{312}L_5 & 0 & Q_{312}L_6 & 0 \\
* & * & * & * & -4h_1(L_2+L_2^T) & 0 & 0 & P_2^T(h_1+1) & 0 & -2h_1L_2 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & Q_{66} & 0 & Q_{68} & 0 & 0 & 0 & 0 & -2Q_{211}L_4 & 0 & 0 \\
* & * & * & * & * & * & Q_{77} & Q_{78} & 0 & 0 & 0 & 0 & 0 & -2Q_{312}L_6 & 0 \\
* & * & * & * & * & * & * & Q_{88} & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{-k_o/2}G \\
* & * & * & * & * & * & * & * & -N_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -3N_1 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & -N_2 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & -N_2 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & -3N_2 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & -3N_2 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & -k_qG
\end{bmatrix} < 0. \quad (60)$$

By Lemma 2, (57) is expressed as

$$\boldsymbol{\Pi}_0(d(u))+\varepsilon_0^{-1}\bar{\boldsymbol{H}}_0\bar{\boldsymbol{H}}_0^T+\varepsilon_0\bar{\boldsymbol{E}}_0^T\bar{\boldsymbol{E}}_0<\mathbf{0} \tag{61}$$

where $\varepsilon_0>0$. Now, with the help of Schur's complement, (61) is written as

$$\left[\begin{array}{cccccccccccccccc}
\boldsymbol{Q}_{11}+\varepsilon_0\boldsymbol{E}_0^T\boldsymbol{E}_0 & \boldsymbol{Q}_{12} & -\boldsymbol{P}_7+\boldsymbol{P}_4 & \mathbf{0} & \boldsymbol{Q}_{15} & \boldsymbol{Q}_{16} & \boldsymbol{Q}_{17} & \boldsymbol{Q}_{18} & h_1\boldsymbol{L}_1 & h_1\boldsymbol{L}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{Q}_{115} & \mathbf{0}\\
* & \boldsymbol{Q}_{22} & \boldsymbol{Q}_{23} & \boldsymbol{P}_7 & \boldsymbol{Q}_{25} & \boldsymbol{Q}_{26} & \boldsymbol{Q}_{27} & -\boldsymbol{P}_2^T & -h_1\boldsymbol{L}_1 & h_1\boldsymbol{L}_2 & \mathcal{Q}_{211}\boldsymbol{L}_3 & \mathbf{0} & \mathcal{Q}_{211}\boldsymbol{L}_4 & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
* & * & \boldsymbol{Q}_{33} & \boldsymbol{Q}_{34} & \boldsymbol{Q}_{35} & \boldsymbol{Q}_{36} & \boldsymbol{Q}_{37} & -\boldsymbol{P}_3^T+k_q\bar{\boldsymbol{A}}_d^T\boldsymbol{G} & \mathbf{0} & \mathbf{0} & -\mathcal{Q}_{211}\boldsymbol{L}_3 & \mathcal{Q}_{312}\boldsymbol{L}_5 & \mathcal{Q}_{211}\boldsymbol{L}_4 & \mathcal{Q}_{312}\boldsymbol{L}_6 & -k_q\sqrt{-2k_o}\bar{\boldsymbol{A}}_d^T\boldsymbol{G} & \mathbf{0}\\
* & * & * & \boldsymbol{Q}_{44} & -\boldsymbol{P}_7^T(h_1+1) & \boldsymbol{Q}_{46} & \boldsymbol{Q}_{47} & -\boldsymbol{P}_4^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathcal{Q}_{312}\boldsymbol{L}_5 & \mathbf{0} & \mathcal{Q}_{312}\boldsymbol{L}_6 & \mathbf{0} & \mathbf{0}\\
* & * & * & * & -4h_1(\boldsymbol{L}_2+\boldsymbol{L}_2^T) & \mathbf{0} & \mathbf{0} & \boldsymbol{P}_2^T(h_1+1) & \mathbf{0} & -2h_1\boldsymbol{L}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
* & * & * & * & * & \boldsymbol{Q}_{66} & \mathbf{0} & \boldsymbol{Q}_{68} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -2\mathcal{Q}_{211}\boldsymbol{L}_4 & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
* & * & * & * & * & * & \boldsymbol{Q}_{77} & \boldsymbol{Q}_{78} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -2\mathcal{Q}_{312}\boldsymbol{L}_6 & \mathbf{0} & \mathbf{0}\\
* & * & * & * & * & * & * & \boldsymbol{Q}_{88} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \sqrt{-k_o/2}\,\boldsymbol{G} & k_q\boldsymbol{G}\boldsymbol{H}_0\\
* & * & * & * & * & * & * & * & -\boldsymbol{N}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
* & * & * & * & * & * & * & * & * & -3\boldsymbol{N}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
* & * & * & * & * & * & * & * & * & * & -\boldsymbol{N}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
* & * & * & * & * & * & * & * & * & * & * & -\boldsymbol{N}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
* & * & * & * & * & * & * & * & * & * & * & * & -3\boldsymbol{N}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
* & * & * & * & * & * & * & * & * & * & * & * & * & -3\boldsymbol{N}_2 & \mathbf{0} & \mathbf{0}\\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & -k_q\boldsymbol{G} & -k_q\sqrt{-2k_o}\boldsymbol{G}\boldsymbol{H}_0\\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & -\varepsilon_0\boldsymbol{I}
\end{array}\right]<\mathbf{0}, \tag{62}$$

Following the steps similar to (57) to (62), one can easily show that (62) is equivalent to $\boldsymbol{\Psi}(d(u))<\mathbf{0}$. Observe that, with respect to $d(u)$, $\boldsymbol{\Psi}(d(u))$ is an affine function and by the feature of an affine function, $\boldsymbol{\Psi}(d(u))<\mathbf{0}$ holds when (8a) and (8b) satisfy simultaneously. Now, the proof of Theorem 1 is completed.

Next, for the case of constant delay i.e., $h_1=h_2=h$, the variable time-lagged system (1) becomes

$$\chi(u+1)=\boldsymbol{O}\{\boldsymbol{Q}(\gamma(u))\}=$$
$$\lambda((\boldsymbol{A}+\Delta\boldsymbol{A})\chi(u)+(\boldsymbol{A}_d+\Delta\boldsymbol{A}_d)\chi(u-h)). \tag{63}$$

Pertaining to the system (63) with (2) and (4), we will achieve the following result.

Corollary 1. The system (63), (2), (4) is GAS if there exist matrices $\mathbf{0}<\bar{\boldsymbol{P}}^T=\bar{\boldsymbol{P}}=\begin{bmatrix}\bar{\boldsymbol{P}}_1 & \bar{\boldsymbol{P}}_2\\ * & \bar{\boldsymbol{P}}_3\end{bmatrix}$, $\mathbf{0}<\boldsymbol{Q}^T=\boldsymbol{Q}$, $\mathbf{0}<\boldsymbol{R}^T=\boldsymbol{R}$, a matrix (diagonal) $\mathbf{0}<\boldsymbol{G}^T=\boldsymbol{G}$, any matrices $\boldsymbol{L}_1$, $\boldsymbol{L}_2$ with suitable dimensions and two positive scalars $\boldsymbol{\varepsilon}_0$, $\boldsymbol{\varepsilon}_1$ such that (64) holds.

$$\begin{bmatrix} \boldsymbol{S}_{11}+\varepsilon_0\boldsymbol{E}_0^{\ T}\boldsymbol{E}_0 & S_{12} & \boldsymbol{S}_{13} & k_q\boldsymbol{A}^T\boldsymbol{G}-h^2\boldsymbol{R} & h\boldsymbol{L}_1 & h\boldsymbol{L}_2 & -k_q\sqrt{-2k_o}\boldsymbol{A}^T\boldsymbol{G} & \mathbf{0} & \mathbf{0} \\ * & \boldsymbol{S}_{22}+\varepsilon_1\boldsymbol{E}_1^{\ T}\boldsymbol{E}_1 & \boldsymbol{S}_{23} & -\bar{\boldsymbol{P}}_2^{\ T}+k_q\boldsymbol{A}_d^{\ T}\boldsymbol{G} & h\boldsymbol{L}_1 & h\boldsymbol{L}_2 & -k_q\sqrt{-2k_o}\boldsymbol{A}_d^{\ T}\boldsymbol{G} & \mathbf{0} & \mathbf{0} \\ * & * & -4h(\boldsymbol{L}_2+\boldsymbol{L}_2^{\ T}) & \bar{\boldsymbol{P}}_2^{\ T}(h+1) & \mathbf{0} & -2h\boldsymbol{L}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & \boldsymbol{S}_{44} & \mathbf{0} & \mathbf{0} & \sqrt{-k_o/2}\boldsymbol{G} & k_q\boldsymbol{G}\boldsymbol{H}_0 & k_q\boldsymbol{G}\boldsymbol{H}_1 \\ * & * & * & * & -\boldsymbol{R} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & * & -3\boldsymbol{R} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & * & * & -k_q\boldsymbol{G} & -k_q\sqrt{-2k_o}\boldsymbol{G}\boldsymbol{H}_0 & -k_q\sqrt{-2k_o}\boldsymbol{G}\boldsymbol{H}_1 \\ * & * & * & * & * & * & * & -\varepsilon_0\boldsymbol{I} & \mathbf{0} \\ * & * & * & * & * & * & * & * & -\varepsilon_1\boldsymbol{I} \end{bmatrix} < \mathbf{0} \quad (64)$$

where

$$\boldsymbol{S}_{11}=h^2\boldsymbol{R}-\bar{\boldsymbol{P}}_3-\bar{\boldsymbol{P}}_1+\bar{\boldsymbol{P}}_2+\bar{\boldsymbol{P}}_2^{\ T}+\boldsymbol{Q}-h(\boldsymbol{L}_1+\boldsymbol{L}_1^{\ T})-h(\boldsymbol{L}_2+\boldsymbol{L}_2^{\ T}), \quad (65a)$$

$$\boldsymbol{S}_{22}=\bar{\boldsymbol{P}}_3-\boldsymbol{Q}-h(\boldsymbol{L}_2+\boldsymbol{L}_2^{\ T})-h(\boldsymbol{L}_2+\boldsymbol{L}_2^{\ T}), \quad (65b)$$

$$S_{12}=h(\boldsymbol{L}_1+\boldsymbol{L}_1^{\ T})-h(\boldsymbol{L}_2+\boldsymbol{L}_2^{\ T}), \quad (65c)$$

$$\boldsymbol{S}_{13}=\bar{\boldsymbol{P}}_3(h+1)-\bar{\boldsymbol{P}}_2(h+1)+2h(\boldsymbol{L}_2+\boldsymbol{L}_2^{\ T}), \quad (65d)$$

$$\boldsymbol{S}_{23}=-\bar{\boldsymbol{P}}_5(h+1)+2h(\boldsymbol{L}_2+\boldsymbol{L}_2^{\ T}), \quad (65e)$$

$$\boldsymbol{S}_{44}=\bar{\boldsymbol{P}}_1+h^2\boldsymbol{R}+((k_o/2k_q)-2)\boldsymbol{G}. \quad (65f)$$

Proof. By considering the Lyapunov function

$$V(\chi(u))=\eta^T(u)\bar{\boldsymbol{P}}\eta(u)+\sum_{s=u-h}^{u-1}\chi^T(s)\boldsymbol{Q}\,\chi(s) + h\sum_{\theta=-h}^{-1}\sum_{s=u+\theta}^{u-1}\Gamma^T(s)\boldsymbol{R}\Gamma(s) \quad (66a)$$

$$\eta^T(u)=[\chi^T(u)\quad \sum_{s=u-h}^{u-1}\chi^T(s)] \quad (66b)$$

and following the steps used in the proof of Theorem 1, one can easily be arrived at Corollary 1. The details of the derivation of Corollary 1 are, therefore, omitted. This finishes the proof of Corollary 1.

Remark 2 It may be mentioned that a delay-dependent criterion for the stability of the system (1)-(4) with no parameter uncertainties and nonlinearities (i.e. $\Delta\boldsymbol{A}=\Delta\boldsymbol{A}_d=\mathbf{0}$ and $\boldsymbol{\lambda}(\gamma(u))=\gamma(u)$) is reported by Wu et al. [26, Theorem 1]. A close observation reveals that the approach presented in this paper may be considered as an extension of the criterion given in [26, Theorem 1] for systems with variable time-lag to a model that includes, in addition, parameter uncertainties and quantization/overflow nonlinearities.

Remark 3 Note that the conditions (8) and (64) are dependent in terms of k_o and k_q. Hence, the presented approach may also be useful to determine the values of k_o and k_q (i.e., to find different combinations of quantization and overflow) that is required to assure the stability of the considered system.

Remark 4 By choosing the free matrices $\boldsymbol{L}_1=\dfrac{\boldsymbol{R}}{(a_2-a_1)}$ and $\boldsymbol{L}_2=\dfrac{3\boldsymbol{R}}{(a_2-a_1)}$ considered in Lemma 1, the inequality (5) yields

$$-(a_2-a_1)\sum_{s=u-a_2}^{u-a_1-1}\Gamma^T(s)\boldsymbol{R}\Gamma(s)\le-\begin{bmatrix}\boldsymbol{k}_1\\ \boldsymbol{k}_2\end{bmatrix}^T\begin{bmatrix}\boldsymbol{R} & \mathbf{0}\\ \mathbf{0} & 3\boldsymbol{R}\end{bmatrix}\begin{bmatrix}\boldsymbol{k}_1\\ \boldsymbol{k}_2\end{bmatrix} \quad (67)$$

which is identical to the Wirtinger-based inequality. Note that the inequality (5) of Lemma 1 is more relaxed than that of (67). In view of the above and from Remark 6 of [6], it may be concluded that by employing Lemma 1, one may achieve less conservative results than employing Wirtinger-based inequality [6, 34] and Jensen inequality [5]

Remark 5 In the derivation of Theorem 1, six $n\times n$ matrices $\boldsymbol{L}_i$ (i = 1, 2, ..., 6) are introduced for obtaining less conservative results. However, One can reduce the number of decision variables by selecting the matrices $\boldsymbol{L}_i$ (i = 1, 2, ..., 6) as diagonal matrices.

Remark 6 The conditions proposed in this paper are LMI-based and can be easily tractable using LMI solver of MATLAB [37] along with YALMIP [39].

4 Numerical Examples

The applicability of the proposed conditions is shown by the following numerical examples.

Example 1. Choose the system (1) - (4) with

$$\boldsymbol{A}=\begin{bmatrix}0.8 & 0\\ 0.05 & 0.9\end{bmatrix}, \boldsymbol{A}_d=\begin{bmatrix}-0.1 & 0\\ -0.2 & -0.1\end{bmatrix} \quad (68a)$$

$$\boldsymbol{H}_0=\boldsymbol{H}_1=\begin{bmatrix}0\\ 0.1\end{bmatrix}, \boldsymbol{E}_0=\begin{bmatrix}0.01 & 0\end{bmatrix}, \boldsymbol{E}_1=\begin{bmatrix}0 & 0.01\end{bmatrix} \quad (68b)$$

and the nonlinearities belongs to the sector $[k_o, k_q]=[0 \;\; 1]$, which comprises zeroing, saturation, magnitude truncation, a combination of zeroing and magnitude truncation, a combination of saturation and magnitude truncation, and so on. In [4-6], this example has been considered. Table 1 shows the upper lag limit (h_2) for several given lower lag limits (h_1). In Table 1, while showing the upper lag limit h_2 using Theorem 1 [5], it is assumed that the lower lag limit h_1 is divided into m number of partitions such that $h_1 = \tau m$ where integer τ is partition size. From Table 1, it is also clear that the Theorem 1 provides less conservative results than the previous results [4-6].

Table 1.Upper lag limit h_2 for different lower lag limit h_1 for the system taken in Example 1.

Methods/h_1	2	6	8	12
Theorem 1 [4]	8	8	10	13
Theorem 1 [5]	8 (m=2, τ=1)	9 (m=3, τ=2)	10 (m=4, τ=2)	13 (m=6, τ=2)
Theorem 1 [6]	9	10	11	14
Theorem 1	9	11	12	15

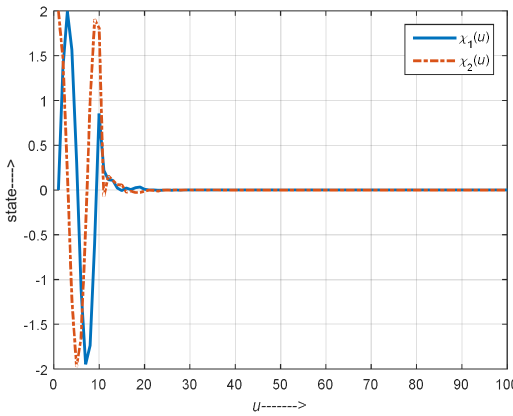

Fig. 1. State trajectories of the system considered in Example 1.

The state trajectories of the chosen system are depicted in Figure 1 where $\boldsymbol{F}_0 = \boldsymbol{F}_1 = 1$, $2 \le d(u) \le 9$ and initial condition is selected randomly. Moreover, in Figure 1, $\chi(u) \to \mathbf{0}$ as $u \to \infty$ implying the global asymptotic stability of the considered system.

Example 2. Choose the system (63) together with

$$\boldsymbol{A}=\begin{bmatrix}0.8 & 0\\ 0 & 0.91\end{bmatrix}, \boldsymbol{A}_d=\begin{bmatrix}-0.1 & 0\\ -0.1 & -0.1\end{bmatrix} \quad (69a)$$

$$\boldsymbol{H}_0=\boldsymbol{H}_1=\begin{bmatrix}0\\ 0.1\end{bmatrix}, \boldsymbol{E}_0=\begin{bmatrix}0.01 & 0\end{bmatrix}, \boldsymbol{E}_1=\begin{bmatrix}0 & 0.01\end{bmatrix} \quad (69b)$$

and $[k_o,\, k_q]=[0 \;\; 1]$. Using Corollary 1, it is verified that the system under consideration is GAS over the delay h = 18.

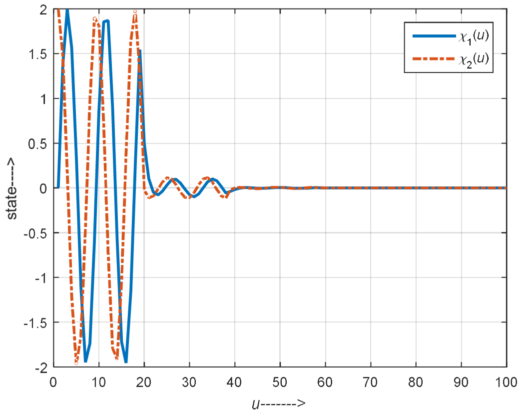

Fig. 2. State trajectories of the system considered in Example 2.

In Figure 2, the state trajectories of the present system are depicted where initial condition is assumed arbitrarily and $\boldsymbol{F}_0 = \boldsymbol{F}_1 = 1$. Further, in Figure 2, $\chi(u) \to \mathbf{0}$ as $u \to \infty$ which implies the global asymptotic stability of the present system.

5 Conclusion

This paper establishes delay-dependent LMI-based stability conditions for the discrete-time uncertain systems having combination of quantization and overflow nonlinearities and variable time-lags. The approach given in this paper, relative to previous approaches, helps in reducing conservativeness which in turn provides a larger stability region. Numerical examples are also given for proving the efficacy of the proposed conditions.

6 Future Scope

The approach presented in this paper can be extended to derive stability criteria for a category of discrete-time systems involving external disturbances, in addition to nonlinearities and variable time-lags which require further investigation. The possible utilization of the present idea for studying the stability analysis of uncertain two-dimensional systems with nonlinearities and variable time-lags appears to be more practical and challenging problem for future work.

Conflict of Interest: NIL

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