Stability of Uncertain 2-*D* Systems with Time-Varying Delays and Quantization/Overflow

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ABSTRACT

In the implementation of two-dimensional (2-*D*) discrete systems (DSs) using computer or digital hardware, quantization and overflow nonlinearities are frequently introduced. These nonlinearities may lead the system towards instability. Apart from these nonlinearities, the presence of time delays and uncertainties in 2-*D* systems leads to system instability. By employing the Lyapunov method, the problem of global asymptotic stability (GAS) of uncertain 2-*D* DSs based on the Roesser model with time-varying delays and different concatenations of quantization and overflow is studied in this paper. To tackle the sum terms involved in the forward difference of the Lyapunov functional, reciprocal convex inequality along with Wirtinger-based inequality techniques are employed. To test the GAS of the DS, a new delay-dependent criterion is derived. The importance of the proposed results is illustrated with the help of numerical examples together with the simulation results.

Keywords: 2-*D* system, delayed system, finite wordlength nonlinearity, global asymptotic stability, uncertain system.

1. INTRODUCTION

During recent decades, significant interest has been devoted to the analysis of two-dimensional (2-*D*) discrete systems (DSs). The 2-*D* DSs are used in various areas, including digital control, mapping in landslide areas, thermal processes, optical fiber networks, 2-*D* controller design, river pollution modeling, gas filtration processes and robot navigation [1-4]. The problems associated with the stability of 2-*D* DSs, such as $l_2 - l_\infty$ stability [5], H_∞ control [6], guaranteed cost control [7], etc., have received a lot of attention.

While realizing DSs using special purpose digital circuit or general purpose computer, one usually face the finite wordlength nonlinearities (FWNs) such as overflow and quantization. These FWNs lead the system towards instability [8]. The result concerning the stability of 2-*D* systems under quantization effects has been reported in [9]. The saturation overflow stability results for 2-*D* DSs have been presented in [8, 10-16]. The stability testing for 2-*D* systems under both quantization and overflow emerges to be more practical [17, 18].

Time delay is one more cause of poor performance and instability in the system. It may occur due to measurement lags, computational delays, limited speed of data transmission and information processing in many parts of the system, etc. Time delay may exist in several realistic systems such as biological systems, communication networks, image processing [19, 20], etc. Many researchers have analyzed the stability of 2-*D* systems using time-varying delay (TVD). Delay-independent results [7, 19, 20] are generally more conservative than delay-dependent results [6, 10, 17, 18, 21-23].

Parameter uncertainties are considered as another factor for producing system instability. They occur due to the finite resolution of the measuring equipment, modeling errors, alterations of the system parameters or several ignored factors. Several publications [6, 18, 21, 23] analyzed the stability of 2-*D* uncertain systems.

In [16], the global asymptotic stability (GAS) of 2-*D* uncertain DSs with saturation overflow nonlinearities and TVDs has been studied. The criterion in [16] applies to saturation and it can not handle the combined effects of quantization and overflow. Since a practical DS functions under the combined impacts of overflow and quantization, the stability of such systems is a crucial issue in practice.

The problem of deriving delay-dependent stability criteria for 2-*D* uncertain DSs represented by Roesser model [24] with TVDs and quantization/overflow nonlinearities is an important and challenging task. By utilizing Jensen-based inequality (JBI) and reciprocal convex inequality (RCI), a delay-dependent stability result for such systems has been established in [18]. However, the approach in [18] is still restrictive. Therefore, there is still enough scope to improve the result in [18].

Motivated by the above discussions, in this paper, we investigate the GAS of uncertain 2-*D* Roesser model [24] with TVDs and quantization/overflow. The primary contributions are as follows.

- The system under study includes a wider class of practical 2-*D* DSs involving FWNs, TVDs and parameter uncertainties.
- A new GAS result for the 2-*D* DS is established by using Wirtinger-based inequality (WBI) and RCI approaches.
- The proposed GAS criterion is shown to be more relaxed than [18].

The paper is structured as follows. The mathematical model of the system and some preliminary results are presented in Section 2. Section 3 reviews previous relevant studies. Section 4 establishes a new GAS criterion. Examples are given in Section 5 to demonstrate the superiority of the presented results. A discussion on the presented method is given in Section 6. Section 7 provides conclusion.

Notations: In this paper, $\mathbb{R}^{p}$ ($\mathbb{R}^{p\times q}$) is the set of real $p\times 1$ ($p\times q$) vectors (matrices); Z_{+} is the set of nonnegative integers; $\boldsymbol{I}$ and $\mathbf{0}$ denote identity and null matrices, respectively; $\mathcal{F}\geq\mathbf{0}$ ($>\mathbf{0}$) implies that $\mathcal{F}$ is a symmetric positive semidefinite (positive definite) matrix; $\mathcal{F}<\mathbf{0}$ implies that

$\mathcal{F}$ is a symmetric negative definite matrix; $\mathcal{F}_1 \oplus \mathcal{F}_2$ stands for $\begin{bmatrix} \mathcal{F}_1 & \mathbf{0} \\ \mathbf{0} & \mathcal{F}_2 \end{bmatrix}$; $\lfloor \gamma \rceil$ represents the closest integer to γ; $*$ indicates the symmetric entries in a symmetric matrix; $\|\cdot\|$ is any matrix or vector norm; sup $\{\cdot\}$ is the supremum of a set.

2. SYSTEM DESCRIPTION

Consider a DS in the setting of 2-*D* Roesser model [24] operating under FWNs, TVDs and parameter uncertainties. The system undertaken is represented by

$$\wp_{11}(\mu,\ \nu) = \begin{bmatrix} \wp^h(\mu+1,\ \nu) \\ \hdashline \wp^v(\mu,\ \nu+1) \end{bmatrix} = \boldsymbol{O}\{\boldsymbol{Q}(\boldsymbol{\sigma}(\mu,\ \nu))\} = \boldsymbol{f}(\boldsymbol{\sigma}(\mu,\ \nu)) = \begin{bmatrix} \boldsymbol{f}^h(\boldsymbol{\sigma}^h(\mu,\ \nu)) \\ \hdashline \boldsymbol{f}^v(\boldsymbol{\sigma}^v(\mu,\ \nu)) \end{bmatrix}, \tag{1a}$$

$$\boldsymbol{f}^h(\boldsymbol{\sigma}^h(\mu,\ \nu)) = \begin{bmatrix} f_1^h(\sigma_1^h(\mu,\ \nu)) & f_2^h(\sigma_2^h(\mu,\ \nu)) & \cdots & f_m^h(\sigma_m^h(\mu,\ \nu)) \end{bmatrix}^{\mathrm{T}}, \tag{1b}$$

$$\boldsymbol{f}^v(\boldsymbol{\sigma}^v(\mu,\ \nu)) = \begin{bmatrix} f_1^v(\sigma_1^v(\mu,\ \nu)) & f_2^v(\sigma_2^v(\mu,\ \nu)) & \cdots & f_n^v(\sigma_n^v(\mu,\ \nu)) \end{bmatrix}^{\mathrm{T}}, \tag{1c}$$

$$\boldsymbol{\sigma}(\mu,\ \nu) = \begin{bmatrix} \boldsymbol{\sigma}^h(\mu,\ \nu) \\ \hdashline \boldsymbol{\sigma}^v(\mu,\ \nu) \end{bmatrix} = (\boldsymbol{W} + \Delta\boldsymbol{W}) \begin{bmatrix} \wp^h(\mu,\ \nu) \\ \hdashline \wp^v(\mu,\ \nu) \end{bmatrix} + (\boldsymbol{W}_d + \Delta\boldsymbol{W}_d) \begin{bmatrix} \wp^h(\mu - d^h(\mu),\ \nu) \\ \hdashline \wp^v(\mu,\ \nu - d^v(\nu)) \end{bmatrix}, \tag{1d}$$

$$\boldsymbol{W} = \begin{bmatrix} \boldsymbol{W}_{11} & \boldsymbol{W}_{12} \\ \boldsymbol{W}_{21} & \boldsymbol{W}_{22} \end{bmatrix},\ \boldsymbol{W}_d = \begin{bmatrix} \boldsymbol{W}_{d_{11}} & \boldsymbol{W}_{d_{12}} \\ \boldsymbol{W}_{d_{21}} & \boldsymbol{W}_{d_{22}} \end{bmatrix},\ \Delta\boldsymbol{W} = \begin{bmatrix} \Delta\boldsymbol{W}_{11} & \Delta\boldsymbol{W}_{12} \\ \Delta\boldsymbol{W}_{21} & \Delta\boldsymbol{W}_{22} \end{bmatrix}, \Delta\boldsymbol{W}_d = \begin{bmatrix} \Delta\boldsymbol{W}_{d_{11}} & \Delta\boldsymbol{W}_{d_{12}} \\ \Delta\boldsymbol{W}_{d_{21}} & \Delta\boldsymbol{W}_{d_{22}} \end{bmatrix}, \tag{1e}$$

where $\mu \in \mathrm{Z}_+$ and $\nu \in \mathrm{Z}_+$ are the spatial coordinates. The $\wp^h(\mu,\ \nu) \in \mathbb{R}^m$ and $\wp^v(\mu,\ \nu) \in \mathbb{R}^n$ are state vectors in horizontal and vertical directions, respectively. The $\boldsymbol{W}_{11} \in \mathbb{R}^{m\times m}$, $\boldsymbol{W}_{12} \in \mathbb{R}^{m\times n}$, $\boldsymbol{W}_{21} \in \mathbb{R}^{n\times m}$, $\boldsymbol{W}_{22} \in \mathbb{R}^{n\times n}$, $\boldsymbol{W}_{d_{11}} \in \mathbb{R}^{m\times m}$, $\boldsymbol{W}_{d_{12}} \in \mathbb{R}^{m\times n}$, $\boldsymbol{W}_{d_{21}} \in \mathbb{R}^{n\times m}$ and $\boldsymbol{W}_{d_{22}} \in \mathbb{R}^{n\times n}$ are known coefficient matrices. The parametric uncertainties are denoted by $\Delta\boldsymbol{W}_{11}$, $\Delta\boldsymbol{W}_{12}$, $\Delta\boldsymbol{W}_{21}$, $\Delta\boldsymbol{W}_{22}$, $\Delta\boldsymbol{W}_{d_{11}}$, $\Delta\boldsymbol{W}_{d_{12}}$, $\Delta\boldsymbol{W}_{d_{21}}$ and $\Delta\boldsymbol{W}_{d_{22}}$. $\boldsymbol{O}(\cdot)$ is the overflow nonlinearities, $\boldsymbol{Q}(\cdot)$ is the quantization nonlinearities and $\boldsymbol{f}(\cdot)$ represents the concatenation of quantization and overflow. The TVDs along the horizontal and vertical directions are $d^h(\mu)$ and $d^v(\nu)$, respectively. The TVDs are assumed to fulfil

$$0 < d_1^h \ \le\ d^h(\mu)\ \le\ d_2^h,\quad 0 < d_1^v \ \le\ d^v(\nu)\ \le\ d_2^v, \tag{2}$$

where d_1^h (d_2^h) and d_1^v (d_2^v) are lower (upper) bounds on delays along the horizontal and vertical directions, respectively.

The $\boldsymbol{f}(\cdot)$ in (1) is bounded by the sector $[k_o,\ k_q]$ which can be characterized by

$$f_i^h(0)=0,\ k_o\sigma_i^{h^2}(\mu,\ \nu)\le f_i^h(\sigma_i^h(\mu,\ \nu))\sigma_i^h(\mu,\ \nu)\ \le k_q\sigma_i^{h^2}(\mu,\ \nu),\qquad i=1,\ 2,\ ...,\ m,\tag{3a}$$

$$f_i^v(0)=0,\ k_o\sigma_i^{v^2}(\mu,\ \nu)\le f_i^v(\sigma_i^v(\mu,\ \nu))\sigma_i^v(\mu,\ \nu)\ \le k_q\sigma_i^{v^2}(\mu,\ \nu),\qquad i=1,\ 2,\ ...,\ n,\tag{3b}$$

$$k_o=\begin{cases}0, & \text{for saturation or zeroing}\\ -1/3, & \text{for triangular}\\ -1, & \text{for 2's complement,}\end{cases}\qquad k_q=\begin{cases}1, & \text{for magnitude truncation (MT)}\\ 2, & \text{for roundoff.}\end{cases}\tag{3c}$$

The uncertainties are considered to be norm-bounded and take the form

$$\Delta\boldsymbol{W}=\boldsymbol{U}_0\ \boldsymbol{\mathfrak{J}}_0\ \boldsymbol{V}_0,\ \Delta\boldsymbol{W}_d=\boldsymbol{U}_1\ \boldsymbol{\mathfrak{J}}_1\ \boldsymbol{V}_1,\tag{4a}$$

where the known matrices are

$$\boldsymbol{U}_i=\begin{bmatrix}\boldsymbol{U}_i^h\\ \hdashline \boldsymbol{U}_i^v\end{bmatrix},\ \boldsymbol{U}_i^h\in\mathbb{R}^{m\times p_i},\ \boldsymbol{U}_i^v\in\mathbb{R}^{n\times p_i},\ \boldsymbol{V}_i=\begin{bmatrix}\boldsymbol{V}_i^h & \vdots & \boldsymbol{V}_i^v\end{bmatrix},\ \boldsymbol{V}_i^h\in\mathbb{R}^{q_i\times m},\ \boldsymbol{V}_i^v\in\mathbb{R}^{q_i\times n},\quad i=0,\ 1.\tag{4b}$$

The unknown matrix $\boldsymbol{\mathfrak{J}}_i\in\mathbb{R}^{p_i\times q_i}$ satisfies

$$\boldsymbol{I}\ge\boldsymbol{\mathfrak{J}}_i^{\mathrm{T}}\boldsymbol{\mathfrak{J}}_i,\quad i=0,\ 1.\tag{4c}$$

The boundary conditions are described by

$$\wp^h(\mu,\ \nu)=\begin{cases}\boldsymbol{\rho}_{\mu\nu}, & \forall\ M>\nu\ge0,\quad 0\ge\mu\ge-d_2^h\\ \boldsymbol{0}, & \forall\ M\le\nu,\quad 0\ge\mu\ge-d_2^h,\end{cases}$$
$$\wp^v(\mu,\ \nu)=\begin{cases}\boldsymbol{\varrho}_{\mu\nu}, & \forall\ N>\mu\ge0,\quad 0\ge\nu\ge-d_2^v\\ \boldsymbol{0}, & \forall\ N\le\mu,\quad 0\ge\nu\ge-d_2^v,\end{cases}$$
$$\rho_{00}=\varrho_{00},\tag{5}$$

where M and N are two positive integers.

Many practical 2-*D* uncertain systems with FWNs and TVDs can be represented by using (1)-(5). Some examples of such systems are 2-*D* digital filtering [9, 25], 2-*D* control systems using FWNs [17], 2-*D* systems realized using finite register length. In these systems, the effects of quantization and overflow are generally unavoidable during information processing. In practice, the delays occurred during information transmission are mostly time-varying in nature [26]. System given by (1) is quite different from that considered in [16]. Various concatenations of quantization and overflow nonlinearities used in practice can be characterized by (1). By contrast, the effects of quantization have been ignored in [16].

The aim of this study is to develop an improved GAS result for 2-*D* DSs given by (1)-(5) using WBI and RCI approaches.

Next, we present the following preliminaries which are needed to establish the key findings of the paper.

Definition [1]**:** The system given by (1)-(5) is globally asymptotically stable if $\lim_{\ell\to\infty}\aleph_\ell = 0$ where

$$\aleph_\ell = \sup\left\{\left\|\begin{bmatrix}\wp^h(\mu,\ \nu)\\ \wp^v(\mu,\ \nu)\end{bmatrix}\right\| : \mu+\nu=\ell,\quad \mu,\ \nu\ge 1\right\}. \tag{6}$$

Lemma 1 [27]**:** For a given matrix $\boldsymbol{J} > \mathbf{0}$ and integers $\mu,\ y,\ x$ satisfying $\mu \ge y \ge x \ge 0,$ if

$$\boldsymbol{\kappa}(\mu,\ x,\ y) = \begin{cases}\dfrac{1}{y-x}\left[\left(2\sum_{s=\mu-y}^{\mu-x-1}\wp(r)\right)+\wp(\mu-x)-\wp(\mu-y)\right], & y > x,\\ 2\wp(\mu-x), & y = x,\end{cases} \tag{7a}$$

then

$$-(y-x)\sum_{r=\mu-y}^{\mu-x-1}\boldsymbol{\chi}^{\mathrm{T}}(r)\boldsymbol{J}\boldsymbol{\chi}(r) \le -\begin{bmatrix}\boldsymbol{\Lambda}_0\\ \boldsymbol{\Lambda}_1\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{J} & \mathbf{0}\\ * & 3\boldsymbol{J}\end{bmatrix}\begin{bmatrix}\boldsymbol{\Lambda}_0\\ \boldsymbol{\Lambda}_1\end{bmatrix}, \tag{7b}$$

where $\boldsymbol{\chi}(r) = \wp(r+1)-\wp(r)$, $\boldsymbol{\Lambda}_0 = \wp(\mu-x)-\wp(\mu-y)$ and $\boldsymbol{\Lambda}_1 = \wp(\mu-x)+\wp(\mu-y)-\boldsymbol{\kappa}(\mu,\ x,\ y).$ The average value gained by $\wp(\mu)$ over the range $[x, y]$ is $\boldsymbol{\Lambda}_0$. $\boldsymbol{\Lambda}_1$ implies the difference of the mean value of $\wp(\mu)$ and its average over $[x, y]$.

Lemma 2 [28]**:** Let for any vectors $\boldsymbol{X}_1,\ \boldsymbol{X}_2,$ matrices $\boldsymbol{T},\ \boldsymbol{M}$ and non-negative scalars $\gamma_1,\ \gamma_2$ such that

$$\gamma_1+\gamma_2 = 1,\ \begin{bmatrix}\boldsymbol{T} & \boldsymbol{M}\\ * & \boldsymbol{T}\end{bmatrix} \ge \mathbf{0}, \tag{8a}$$

$$\boldsymbol{X}_i = \mathbf{0}\ \text{ if }\ \gamma_i = 0 \qquad (i=1,\ 2). \tag{8b}$$

Then we have

$$-\frac{1}{\gamma_1}\boldsymbol{X}_1^{\mathrm{T}}\boldsymbol{T}\boldsymbol{X}_1-\frac{1}{\gamma_2}\boldsymbol{X}_2^{\mathrm{T}}\boldsymbol{T}\boldsymbol{X}_2 \le -\begin{bmatrix}\boldsymbol{X}_1\\ \boldsymbol{X}_2\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{T} & \boldsymbol{M}\\ * & \boldsymbol{T}\end{bmatrix}\begin{bmatrix}\boldsymbol{X}_1\\ \boldsymbol{X}_2\end{bmatrix}. \tag{8c}$$

Lemma 3 [29]**:** Let $\Lambda,\ \Xi,\ \mathfrak{I}$ and $\boldsymbol{\Gamma}$ be real matrices and $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^{\mathrm{T}}$, then

$$\Lambda\,\mathfrak{I}\,\Xi+\Xi^{\mathrm{T}}\,\mathfrak{I}^{\mathrm{T}}\,\Lambda^{\mathrm{T}}+\boldsymbol{\Gamma} < \mathbf{0}, \tag{9a}$$

$\forall\ \boldsymbol{I} \ge \boldsymbol{\mathfrak{I}}^T\boldsymbol{\mathfrak{I}}$ iff there is a scalar $\in > 0$ fulfilling

$$\in\Xi^{\mathrm{T}}\,\Xi+\in^{-1}\Lambda\,\Lambda^{\mathrm{T}}+\boldsymbol{\Gamma} < \mathbf{0}. \tag{9b}$$

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3. LITERATURE REVIEW

The stability of 2-*D* DSs has been extensively studied [2-18, 20-25, 27-46]. The GAS problems of 2-*D* DSs represented by Roesser model in the presence of FWNs have been studied in [8-11, 13, 16, 31-33, 36] whereas the GAS issues for 2-*D* Fornasini-Marchesini Second Local State-Space (FMSLSS) model employing FWNs have been discussed in [3, 5, 12, 25, 34]. In [5, 13], criteria for the $l_2 - l_\infty$ suppression of overflow oscillation in 2-*D* interfered DSs have been reported. For linear shift-invariant 2-*D* DSs, a sufficient stability condition has been proposed in [30]. In [32], stability conditions have been proposed for the overflow oscillations-free implementation of 2-*D* DSs under quantization and overflow constraints. A stability criterion has been established in [33] to avoid the overflow oscillations in 2-*D* digital filters with saturation. For such filters, the criterion established in [33] not only guarantees overflow stability but also provides robustness against disturbances with a specified H_∞ performance. Stability criteria for uncertain 2-*D* DSs using quantization/overflow and interval-like TVDs have been reported in [34]. In [35], H_∞ filtering issue for 2-*D* DSs using polytopic uncertainties has been studied. For state-delayed 2-*D* positive Roesser model, the sufficient and necessary asymptotic stability conditions have been established in [36] by using linear programming method.

In [37], the feedback control problem for network-based 2-*D* uncertain nonlinear systems with time-varying delays based on the FMSLSS model has been studied. A delay-dependent stability analysis for 2-*D* discrete nonlinear switched FMSLSS model with mixed time-varying delays has been carried out in [38]. With the help of Wirtinger-based inequality and reciprocal convexity methods, criteria for the stability of 2-*D* DSs with saturation nonlinearities and time-varying delays have been reported in [14-16]. Sufficient conditions for finite-time stability and boundedness of 2-*D* positive continuous-discrete Roesser model have been presented in [39]. Stability conditions have been derived in [40] for nonlinear 2-*D* Roesser model by employing state and output feedback topologies. The concept of quantized feedback stabilization has been studied in [41] for nonlinear hybrid stochastic time delay systems. In [42], the problems of robust stability and stabilization of hybrid fractional-order multi-dimensional Roesser model have been analyzed. Delay-independent stability criteria in the form of linear programming have been derived in [43] for 2-*D* positive delayed systems with saturation. With the help of Takagi–Sugeno fuzzy-affine models, an output-feedback sliding-mode control problem has been considered in [44] for nonlinear 2-*D* systems.

To obtain delay-dependent stability results for 2-*D* DSs, multiple techniques [10, 18, 27, 28, 45] have been adopted to tackle the sum terms that appear in the forward difference of Lyapunov function. The RCI generally helps to reduce the number of unknown variables and conservativeness [28]. Criteria derived the WBI method [27] are generally less restrictive than those derived via the JBI method [10, 18]. However, there still leftovers a scope for lessening the conservativeness in the existing approaches [10, 18, 27, 28, 45].

It is clear from the above literature review that the stability of uncertain 2-*D* DSs with FWN and delays is a crucial problem.

4. METHOD

The following result gives a new method for evaluating the GAS of the considered system.

Theorem 1: For given integers d_i^h, d_i^v (i=1, 2) with $d_2^h > d_1^h > 0$ and $d_2^v > d_1^v > 0$, the GAS of DS given by (1)-(5) is ensured if there exist $\mathbf{0} < \boldsymbol{O}^h = \begin{bmatrix} \boldsymbol{O}_1^h & \boldsymbol{O}_2^h & \boldsymbol{O}_3^h \\ * & \boldsymbol{O}_4^h & \boldsymbol{O}_5^h \\ * & * & \boldsymbol{O}_6^h \end{bmatrix} \in \mathbb{R}^{3m\times 3m}$, $\mathbf{0} < \boldsymbol{O}^v = \begin{bmatrix} \boldsymbol{O}_1^v & \boldsymbol{O}_2^v & \boldsymbol{O}_3^v \\ * & \boldsymbol{O}_4^v & \boldsymbol{O}_5^v \\ * & * & \boldsymbol{O}_6^v \end{bmatrix} \in \mathbb{R}^{3n\times 3n}$, $\mathbf{0} < \boldsymbol{E}_i^h \in \mathbb{R}^{m\times m}$, $\mathbf{0} < \boldsymbol{E}_i^v \in \mathbb{R}^{n\times n}$ $(i=1,\ 2,\ 3)$, $\mathbf{0} < \boldsymbol{T}_i^h \in \mathbb{R}^{m\times m}$, $\mathbf{0} < \boldsymbol{T}_i^{\ v} \in \mathbb{R}^{n\times n}$ $(i=1,\ 2)$, $\boldsymbol{H}^h = \begin{bmatrix} \boldsymbol{H}_{11}^h & \boldsymbol{H}_{12}^h \\ \boldsymbol{H}_{21}^h & \boldsymbol{H}_{22}^h \end{bmatrix} \in \mathbb{R}^{2m\times 2m}$, $\boldsymbol{H}^v = \begin{bmatrix} \boldsymbol{H}_{11}^v & \boldsymbol{H}_{12}^v \\ \boldsymbol{H}_{21}^v & \boldsymbol{H}_{22}^v \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$, diagonal matrices $\mathbf{0} < \boldsymbol{G}^h \in \mathbb{R}^{m\times m}$, $\mathbf{0} < \boldsymbol{G}^v \in \mathbb{R}^{n\times n}$ and scalars $0 < \epsilon_0$, $0 < \epsilon_1$ satisfying the inequalities (10)-(12)

$$\begin{bmatrix} \boldsymbol{T}^h & \boldsymbol{H}^h \\ * & \boldsymbol{T}^h \end{bmatrix} \geq \mathbf{0}, \tag{10}$$

$$\begin{bmatrix} \boldsymbol{T}^v & \boldsymbol{H}^v \\ * & \boldsymbol{T}^v \end{bmatrix} \geq \mathbf{0}, \tag{11}$$

$$\boldsymbol{\Phi}(d^h(\mu), d^v(\nu))\big|_{d^h(\mu)=d_1^h,\, d^v(\nu)=d_1^v} < \mathbf{0}, \tag{12a}$$

$$\boldsymbol{\Phi}(d^h(\mu), d^v(\nu))\big|_{d^h(\mu)=d_2^h,\, d^v(\nu)=d_1^v} < \mathbf{0}, \tag{12b}$$

$$\boldsymbol{\Phi}(d^h(\mu), d^v(\nu))\big|_{d^h(\mu)=d_1^h,\, d^v(\nu)=d_2^v} < \mathbf{0}, \tag{12c}$$

$$\boldsymbol{\Phi}(d^h(\mu), d^v(\nu))\big|_{d^h(\mu)=d_2^h,\, d^v(\nu)=d_2^v} < \mathbf{0}, \tag{12d}$$

where

$$\Phi(d^h(\mu),\ d^v(\nu)) = \left[\begin{matrix} \boldsymbol{\Phi}_{11} + \in_0 \boldsymbol{V}_0^{\mathrm{T}}\boldsymbol{V}_0 & \mathbf{0} & ((\boldsymbol{O}_3 - \boldsymbol{O}_2)/2) - 2\boldsymbol{T}_1 & -\boldsymbol{O}_3/2 & \boldsymbol{\Phi}_{15} + 3\boldsymbol{T}_1 \\ * & \boldsymbol{\Phi}_{22} + \in_1 \boldsymbol{V}_1^{\mathrm{T}}\boldsymbol{V}_1 & \boldsymbol{\Phi}_{23} & \boldsymbol{\Phi}_{24} & \mathbf{0} \\ * & * & -\boldsymbol{E}_1 - 4(\boldsymbol{T}_1 + \boldsymbol{T}_2) & \boldsymbol{\Phi}_{34} & \boldsymbol{\Phi}_{35} + 3\boldsymbol{T}_1 \\ * & * & * & -\boldsymbol{E}_2 - 4\boldsymbol{T}_2 & -\boldsymbol{Y}_1(\boldsymbol{O}_5^{\mathrm{T}}/2) \\ * & * & * & * & -3\boldsymbol{T}_1 \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{matrix}\right.$$

$$\left.\begin{array}{cccccc}
\boldsymbol{\Phi}_{16} & \boldsymbol{\Phi}_{17} & \boldsymbol{\Phi}_{18}+(\boldsymbol{O}_2^{\mathrm{T}}/2)+k_q\boldsymbol{W}^{\mathrm{T}}\boldsymbol{G} & \boldsymbol{\Phi}_{19} & \mathbf{0} & \mathbf{0} \\
\boldsymbol{\Phi}_{26} & \boldsymbol{\Phi}_{27} & k_q\boldsymbol{W}_d^{\mathrm{T}}\boldsymbol{G} & \boldsymbol{\Phi}_{29} & \mathbf{0} & \mathbf{0} \\
\boldsymbol{\Phi}_{36}+3\boldsymbol{T}_2 & \boldsymbol{\Phi}_{37} & (\boldsymbol{O}_3^{\mathrm{T}}-\boldsymbol{O}_2^{\mathrm{T}})/2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\boldsymbol{\Phi}_{46} & \boldsymbol{\Phi}_{47} & -\boldsymbol{O}_3^{\mathrm{T}}/2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
0 & 0 & \boldsymbol{Y}_1(\boldsymbol{O}_2^{\mathrm{T}}/2) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
-3\boldsymbol{T}_2 & -\boldsymbol{H}_{22} & \boldsymbol{Y}_2(\boldsymbol{O}_3^{\mathrm{T}}/2) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & -3\boldsymbol{T}_2 & \boldsymbol{Y}_3(\boldsymbol{O}_3^{\mathrm{T}}/2) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & \boldsymbol{\Phi}_{88} & \sqrt{\dfrac{-k_o}{2}}\boldsymbol{G} & k_q\boldsymbol{G}\boldsymbol{U}_0 & k_q\boldsymbol{G}\boldsymbol{U}_1 \\
* & * & * & -k_q\boldsymbol{G} & -k_q\sqrt{-2k_o}\boldsymbol{G}\boldsymbol{U}_0 & -k_q\sqrt{-2k_o}\boldsymbol{G}\boldsymbol{U}_1 \\
* & * & * & * & -\in_0\boldsymbol{I} & \mathbf{0} \\
* & * & * & * & * & -\in_1\boldsymbol{I}
\end{array}\right],$$

$\boldsymbol{\Phi}_{11} = -\boldsymbol{O}_1+((\boldsymbol{O}_2+\boldsymbol{O}_2^{\mathrm{T}})/2)-4\boldsymbol{T}_1+\sum_{i=1}^{3}\boldsymbol{E}_i+\boldsymbol{Y}_4\boldsymbol{E}_3-\boldsymbol{\Phi}_{18}$, $\boldsymbol{\Phi}_{15}=\boldsymbol{Y}_1(\boldsymbol{O}_4-\boldsymbol{O}_2)/2$, $\boldsymbol{\Phi}_{16}=\boldsymbol{Y}_2(\boldsymbol{O}_5-\boldsymbol{O}_3)/2$,

$\boldsymbol{\Phi}_{17}=\boldsymbol{Y}_3(\boldsymbol{O}_5-\boldsymbol{O}_3)/2$, $\boldsymbol{\Phi}_{18}=-(\boldsymbol{Y}_1^2\boldsymbol{T}_1+\boldsymbol{Y}_4^2\boldsymbol{T}_2)$, $\boldsymbol{\Phi}_{19}=-k_q\sqrt{-2k_o}\boldsymbol{W}^{\mathrm{T}}\boldsymbol{G}$, $\boldsymbol{Y}_1=d_1^h\boldsymbol{I}_{\boldsymbol{m}}\oplus d_1^v\boldsymbol{I}_{\boldsymbol{n}}$, $\boldsymbol{T}^h=\boldsymbol{T}_2^h\oplus 3\boldsymbol{T}_2^h$,

$\boldsymbol{Y}_2=(d^h(\mu)-d_1^h)\boldsymbol{I}_{\boldsymbol{m}}\oplus(d^v(\nu)-d_1^v)\boldsymbol{I}_{\boldsymbol{n}}$, $\boldsymbol{Y}_3=(d_2^h-d^h(\mu))\boldsymbol{I}_{\boldsymbol{m}}\oplus(d_2^v-d^v(\nu))\boldsymbol{I}_{\boldsymbol{n}}$, $\boldsymbol{Y}_4=d_{12}^h\boldsymbol{I}_{\boldsymbol{m}}\oplus d_{12}^v\boldsymbol{I}_{\boldsymbol{n}}$,
$\boldsymbol{T}^v=\boldsymbol{T}_2^v\oplus 3\boldsymbol{T}_2^v$, $\boldsymbol{\Phi}_{22}=-\boldsymbol{E}_3-8\boldsymbol{T}_2+\boldsymbol{H}_{11}+\boldsymbol{H}_{11}^{\mathrm{T}}+\boldsymbol{H}_{12}+\boldsymbol{H}_{12}^{\mathrm{T}}-\boldsymbol{H}_{21}-\boldsymbol{H}_{21}^{\mathrm{T}}-\boldsymbol{H}_{22}-\boldsymbol{H}_{22}^{\mathrm{T}}$,
$\boldsymbol{\Phi}_{23}=-2\boldsymbol{T}_2-\boldsymbol{H}_{11}^{\mathrm{T}}-\boldsymbol{H}_{12}^{\mathrm{T}}-\boldsymbol{H}_{21}^{\mathrm{T}}-\boldsymbol{H}_{22}^{\mathrm{T}}$, $\boldsymbol{\Phi}_{24}=-2\boldsymbol{T}_2-\boldsymbol{H}_{11}+\boldsymbol{H}_{12}+\boldsymbol{H}_{21}-\boldsymbol{H}_{22}$, $\boldsymbol{\Phi}_{26}=3\boldsymbol{T}_2+\boldsymbol{H}_{21}^{\mathrm{T}}+\boldsymbol{H}_{22}^{\mathrm{T}}$,

$\boldsymbol{\Phi}_{27}=3\boldsymbol{T}_2-\boldsymbol{H}_{21}+\boldsymbol{H}_{22}$, $\boldsymbol{\Phi}_{29}=-k_q\sqrt{-2k_o}\boldsymbol{W}_d^{\mathrm{T}}\boldsymbol{G}$, $\boldsymbol{\Phi}_{34}=\boldsymbol{H}_{11}-\boldsymbol{H}_{12}+\boldsymbol{H}_{21}-\boldsymbol{H}_{22}$, $\boldsymbol{\Phi}_{35}=\boldsymbol{Y}_1(\boldsymbol{O}_5^T-\boldsymbol{O}_4)/2$,

$\boldsymbol{\Phi}_{36}=\boldsymbol{Y}_2(\boldsymbol{O}_6-\boldsymbol{O}_5)/2$, $\boldsymbol{\Phi}_{37}=\boldsymbol{Y}_3((\boldsymbol{O}_6-\boldsymbol{O}_5)/2)+\boldsymbol{H}_{12}+\boldsymbol{H}_{22}$, $\boldsymbol{\Phi}_{46}=-\boldsymbol{Y}_2(\boldsymbol{O}_6/2)-\boldsymbol{H}_{12}^{\mathrm{T}}+\boldsymbol{H}_{22}^{\mathrm{T}}$,

$\boldsymbol{\Phi}_{47}=-\boldsymbol{Y}_3(\boldsymbol{O}_6/2)+3\boldsymbol{T}_2$, $\boldsymbol{\Phi}_{88}=\boldsymbol{O}_1-\boldsymbol{\Phi}_{18}+[(k_o/(2k_q))-2]\boldsymbol{G}$, $d_{12}^h=d_2^h-d_1^h$, $d_{12}^v=d_2^v-d_1^v$,

$\boldsymbol{O}_i=\boldsymbol{O}_i^h\oplus\boldsymbol{O}_i^v(i=1,\ 2,\ \ldots,\ 6)$, $\boldsymbol{E}_i=\boldsymbol{E}_i^h\oplus\boldsymbol{E}_i^v(i=1,\ 2,\ 3)$, $\boldsymbol{T}_i=\boldsymbol{T}_i^h\oplus\boldsymbol{T}_i^v(i=1,\ 2)$, $\boldsymbol{G}=\boldsymbol{G}^h\oplus\boldsymbol{G}^v$,
$\boldsymbol{H}_{11}=\boldsymbol{H}_{11}^h\oplus\boldsymbol{H}_{11}^v$, $\boldsymbol{H}_{12}=\boldsymbol{H}_{12}^h\oplus\boldsymbol{H}_{12}^v$, $\boldsymbol{H}_{21}=\boldsymbol{H}_{21}^h\oplus\boldsymbol{H}_{21}^v$ and $\boldsymbol{H}_{22}=\boldsymbol{H}_{22}^h\oplus\boldsymbol{H}_{22}^v$.

The proof of Theorem 1 is presented in *Appendix I*.

Figure 1 displays the flow chart for the proposed method (Theorem 1). This flowchart takes the system parameters (namely, $\boldsymbol{W},\boldsymbol{W}_d,\boldsymbol{U}_0,\boldsymbol{U}_1,\boldsymbol{V}_0,\boldsymbol{V}_1,d_1^h,d_2^h,\ d_1^v,\ d_2^v$) of the system as input. The feasibility of the GAS conditions in Theorem 1 is examined over $d_1^h\le d^h(\mu)\le\ d_2^h$ and $d_1^v\le\ d^v(\nu)\le\ d_2^v$ using MATLAB LMI solver [29] and YALMIP 3.0 [46]. If Theorem 1 yields a feasible solution, the GAS of the system is confirmed over the given delay ranges. If Theorem 1 fails to give a feasible solution, no conclusion on the GAS can be reached.

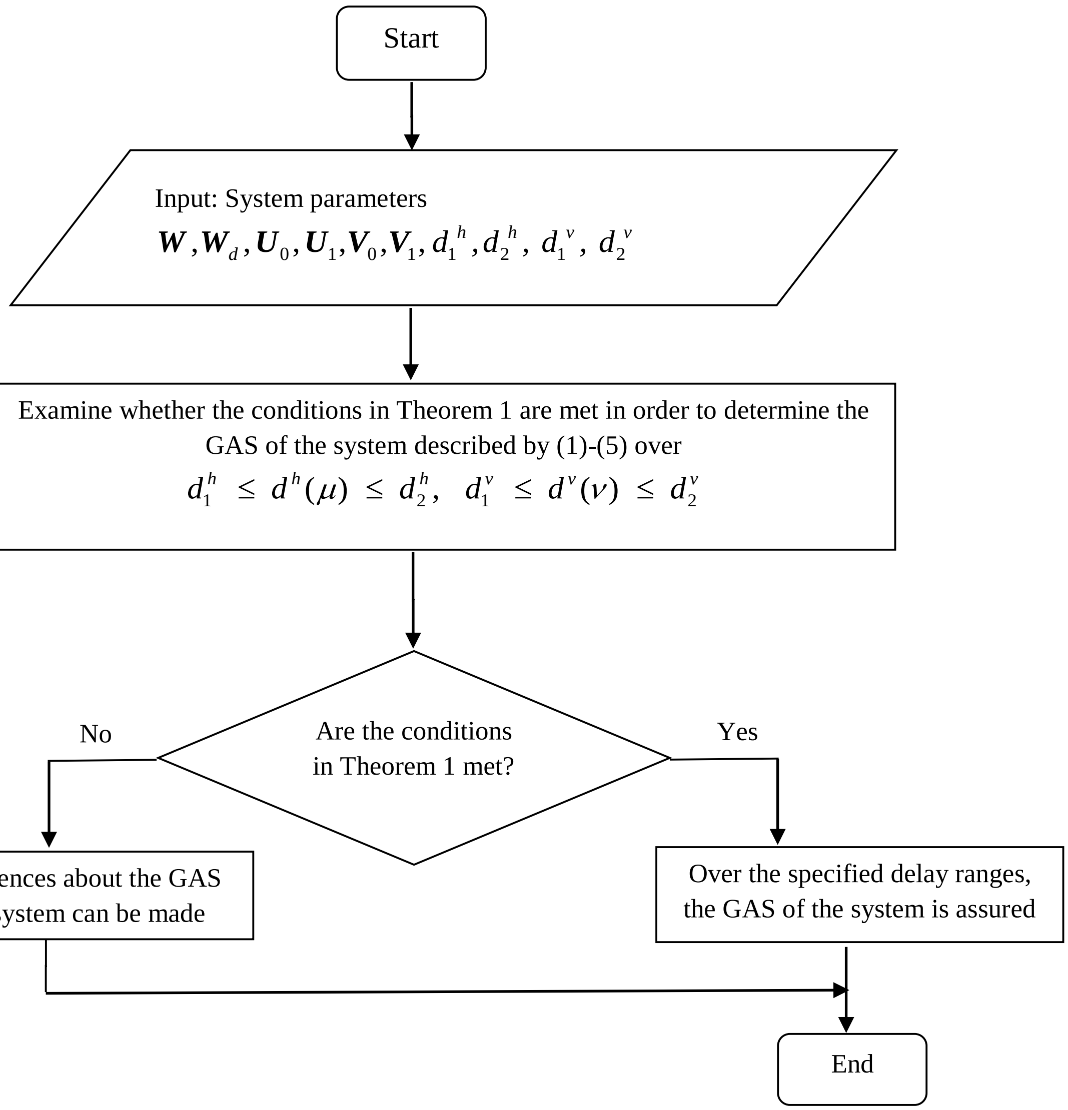

Figure 1 Flowchart for Theorem 1

Remark 1: Pertaining saturation overflow nonlinearities, a criterion has been developed in [16] for evaluating the GAS of a class of 2-*D* uncertain DSs with TVDs. The quantization effects have been ignored in the approach of [16]. It may be highlighted that [16] utilizes the non-negativeness of β where β is given by

$$\beta = \boldsymbol{\sigma}^{\mathrm{T}}(\mu,\ \nu)\boldsymbol{C}\boldsymbol{f}(\boldsymbol{\sigma}(\mu,\ \nu)) + \boldsymbol{f}^{\mathrm{T}}(\boldsymbol{\sigma}(\mu,\ \nu))\boldsymbol{C}^{\mathrm{T}}\boldsymbol{\sigma}(\mu,\ \nu) - \boldsymbol{f}^{\mathrm{T}}(\boldsymbol{\sigma}(\mu,\ \nu))(\boldsymbol{C}+\boldsymbol{C}^{\mathrm{T}})\boldsymbol{f}(\boldsymbol{\sigma}(\mu,\ \nu)) \tag{13}$$

and $\boldsymbol{C}$ is a (row) diagonally dominant matrix with positive diagonal elements and $\boldsymbol{f}(\boldsymbol{\sigma}(\mu,\ \nu))$ represents the saturation nonlinearities. In contrast, the present approach relies on the non-negativeness δ (see (29)), covering all types of nonlinearities falling within the sector $[k_o,\ k_q]$. The results in [16] are inapplicable to the systems involving various concatenations of overflow and quantization. Thus, the current approach is quite different from [16].

Remark 2: The conditions given in (12) (see Theorem 1) involve the parameters k_o and k_q. These parameters do not depend on the wordlength used to realize the DS given by (1)-(5). Thus, Theorem 1 is also suitable for checking the GAS of 2-*D* system realized with variable wordlength for different signals.

Remark 3: In this paper, the reduced conservatism is achieved by using WBI together with RCI which bound $\boldsymbol{S}_2^h(\mu,\ \nu)$ with $d_1^h \leqslant d^h(\mu) \leqslant d_2^h$ as

$$\boldsymbol{S}_2^h(\mu,\ \nu) \le -\begin{bmatrix} \wp^h(\mu-d_1^h,\ \nu)-\wp^h(\mu-d^h(\mu),\ \nu) \\ \wp^h(\mu-d_1^h,\ \nu)+\wp^h(\mu-d^h(\mu),\ \nu)-\boldsymbol{\chi}(\mu,\ d_1^h,\ d^h(\mu)) \\ \wp^h(\mu-d^h(\mu),\ \nu)-\wp^h(\mu-d_2^h,\ \nu) \\ \wp^h(\mu-d^h(\mu),\ \nu)+\wp^h(\mu-d_2^h,\ \nu)-\boldsymbol{\chi}(\mu,\ d^h(\mu),\ d_2^h) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{T}^h & \boldsymbol{H}^h \\ * & \boldsymbol{T}^h \end{bmatrix} \begin{bmatrix} \wp^h(\mu-d_1^h,\ \nu)-\wp^h(\mu-d^h(\mu),\ \nu) \\ \wp^h(\mu-d_1^h,\ \nu)+\wp^h(\mu-d^h(\mu),\ \nu)-\boldsymbol{\chi}(\mu,\ d_1^h,\ d^h(\mu)) \\ \wp^h(\mu-d^h(\mu),\ \nu)-\wp^h(\mu-d_2^h,\ \nu) \\ \wp^h(\mu-d^h(\mu),\ \nu)+\wp^h(\mu-d_2^h,\ \nu)-\boldsymbol{\chi}(\mu,\ d^h(\mu),\ d_2^h) \end{bmatrix} \quad (14a)$$

with $\begin{bmatrix} \boldsymbol{T}^h & \boldsymbol{H}^h \\ * & \boldsymbol{T}^h \end{bmatrix} \ge \boldsymbol{0}$. In particular, when $\boldsymbol{H}^h = \boldsymbol{0}$ and $d^h(\mu) = d_2^h$, (14a) reduces to

$$\boldsymbol{S}_2^h(\mu,\ \nu) \le -\begin{bmatrix} \wp^h(\mu-d_1^h,\ \nu)-\wp^h(\mu-d^h(\mu),\ \nu) \\ \wp^h(\mu-d_1^h,\ \nu)+\wp^h(\mu-d^h(\mu),\ \nu)-\boldsymbol{\chi}(\mu,\ d_1^h,\ d^h(\mu)) \end{bmatrix}^{\mathrm{T}} \boldsymbol{T}^h \begin{bmatrix} \wp^h(\mu-d_1^h,\ \nu)-\wp^h(\mu-d^h(\mu),\ \nu) \\ \wp^h(\mu-d_1^h,\ \nu)+\wp^h(\mu-d^h(\mu),\ \nu)-\boldsymbol{\chi}(\mu,\ d_1^h,\ d^h(\mu)) \end{bmatrix} \quad (14b)$$

which is obtained via WBI approach. Following Remark 1 [47] and Remark 6 [48], it is easy to conclude that (14b) is less stringent than that obtained via JBI approach.

5. RESULT

To discuss the significance of obtained results, we consider the following examples.

Example 1: Consider the DS defined by (1)-(5) with

$$\boldsymbol{W}_{11} = \begin{bmatrix} 0.6 & -0.32 \\ 0.19 & 0.25 \end{bmatrix},\ \boldsymbol{W}_{12} = \begin{bmatrix} -0.1 \\ 0.54 \end{bmatrix},\ \boldsymbol{W}_{21} = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix},\ \boldsymbol{W}_{22} = 0.16, \quad (15a)$$

$$\boldsymbol{W}_{d_{11}} = \begin{bmatrix} 0.1 & 0.01 \\ 0.11 & 0.05 \end{bmatrix},\ \boldsymbol{W}_{d_{12}} = \begin{bmatrix} 0.03 \\ -0.12 \end{bmatrix},\boldsymbol{W}_{d_{21}} = \begin{bmatrix} 0.02 & 0.06 \end{bmatrix},\ \boldsymbol{W}_{d_{22}} = 0.15, \quad (15b)$$

$$\boldsymbol{U}_0 = \boldsymbol{U}_1 = \begin{bmatrix} 0 \\ 0.1 \\ 0.1 \end{bmatrix},\ \boldsymbol{V}_0 = \begin{bmatrix} 0.01 & 0 & 0 \end{bmatrix},\ \boldsymbol{V}_1 = \begin{bmatrix} 0 & 0.01 & 0 \end{bmatrix}, \quad (15c)$$

$$d_1^v = 2,\ d_2^v = 6,\ k_o = -1,\ k_q = 1. \quad (15d)$$

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This example was also considered in [18]. The nonlinearities in the present DS cover saturation, zeroing, MT, 2's complement overflow, triangular, MT and saturation combinations, MT and zeroing combinations, etc. Table 1 displays the values of d_2^h (which ensures the GAS) obtained via Theorem 1 for some given values of d_1^h. It is apparent from Table 1 that Theorem 1 yields better results than Theorem 1 [18].

Table 1 Upper delay bound d_2^h for various d_1^h in Example 1

Methods/ d_1^h for $2 \le d^v(\nu) \le 6$	3	5	7	9
Theorem 1 [18]	10	12	14	16
Theorem 1 (Proposed)	16	18	20	22

By characterizing the FWNs via the idea of sectors [49], the GAS of the current system is implied by

$$\begin{bmatrix} \wp_1^h(\mu+1,\ \nu) \\ \wp_2^h(\mu+1,\ \nu) \\ \wp^v(\mu,\ \nu+1) \end{bmatrix} = \left\{ \begin{bmatrix} 0.6K_1 & -0.32K_1 & -0.1K_1 \\ 0.19K_2 & 0.25K_2 & 0.54K_2 \\ 0.1K_3 & 0.1K_3 & 0.16K_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0.001\mathfrak{I}_0 K_2 & 0 & 0 \\ 0.001\mathfrak{I}_0 K_3 & 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} \wp_1^h(\mu,\ \nu) \\ \wp_2^h(\mu,\ \nu) \\ \wp^v(\mu,\ \nu) \end{bmatrix}$$

$$+ \left\{ \begin{bmatrix} 0.1K_1 & 0.01K_1 & 0.03K_1 \\ 0.11K_2 & 0.05K_2 & -0.12K_2 \\ 0.02K_3 & 0.06K_3 & 0.15K_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.001\mathfrak{I}_1 K_2 & 0 \\ 0 & 0.001\mathfrak{I}_1 K_3 & 0 \end{bmatrix} \right\} \begin{bmatrix} \wp_1^h(\mu - d^h(\mu),\ \nu) \\ \wp_2^h(\mu - d^h(\mu),\ \nu) \\ \wp^v(\mu,\ \nu - d^v(\nu)) \end{bmatrix}, \qquad (16)$$

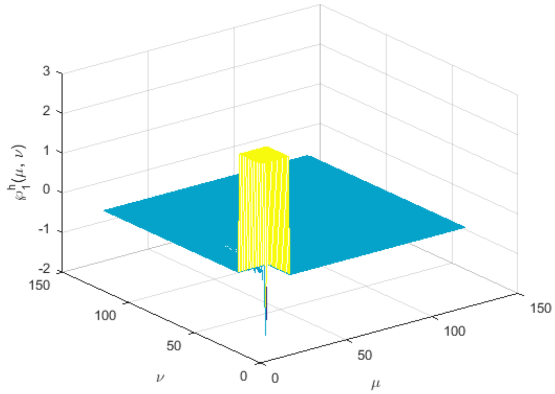
where $K_i \in [-1,\ 1],\ i = 1,\ 2,\ 3.$

For $K_1 = K_2 = -1,\ K_3 = 1,\ \mathfrak{I}_0 = \mathfrak{I}_1 = 1$, $d^h(\mu) = \lfloor |13\sin(\frac{180}{\pi}(\mu-1))| \rceil + 3$ and $d^v(\nu) = \lfloor | 4\sin(\frac{180}{\pi}(\nu-1)) | \rceil + 2$, the state trajectories of the current DS are depicted in Figure 2. The plots of TVDs used in the simulation for this example are depicted in Figure 3. To study the behaviour of state trajectories, the initial conditions are selected as

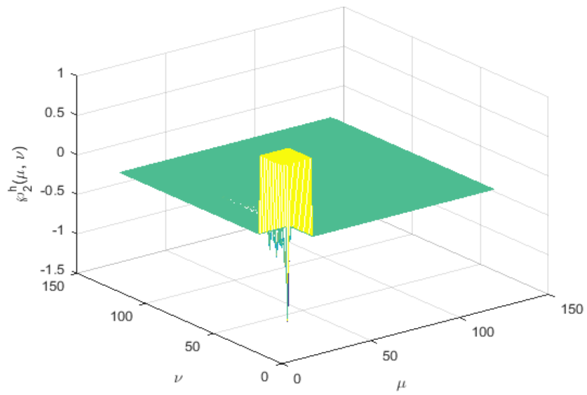
$$\boldsymbol{\wp}^h(\mu,\ \nu) = \begin{bmatrix} \wp_1^h(\mu,\ \nu) & \wp_2^h(\mu,\ \nu) \end{bmatrix}^{\mathrm{T}} = \begin{cases} \begin{bmatrix} -3 & -1 \end{bmatrix}^{\mathrm{T}}, & \forall\ 17 > \nu \ge 0, \quad 16 \ge \mu \ge 0, \\ \mathbf{0}, & \forall\ 17 \le \nu, \quad 16 \ge \mu \ge 0, \end{cases} \qquad (17a)$$

$$\wp^v(\mu,\ \nu) = \begin{cases} 2, & \forall\ 17 > \mu \ge 0, \quad 6 \ge \nu \ge 0, \\ 0, & \forall\ 17 \le \mu, \quad 6 \ge \nu \ge 0. \end{cases} \qquad (17b)$$

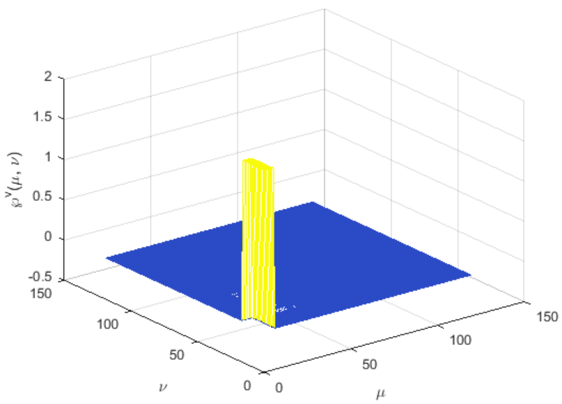
Figure 2 shows that the system state vector tends to zero as $\mu + \nu \to \infty$. This supports the GAS of the present DS.



(a)

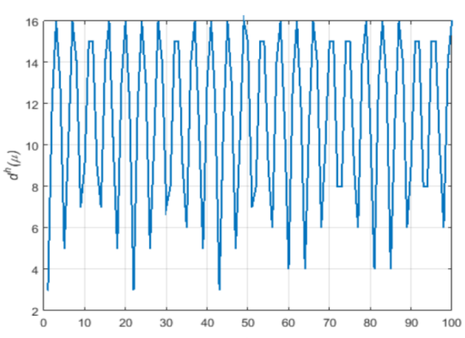


(b)

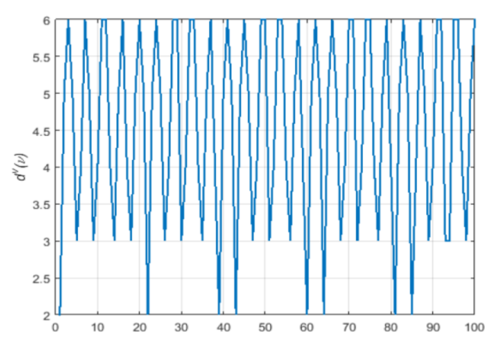


(c)

Figure 2. State trajectories for the system in Example 1.



(a)



(b)

Figure 3. TVDs used in the simulation for Example 1.

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Example 2: Consider the DS given by (1)-(5) with

$$\boldsymbol{W}_{11}=\begin{bmatrix}0.54 & -0.29\\ 0.2 & 0.25\end{bmatrix},\ \boldsymbol{W}_{12}=\begin{bmatrix}-0.1 & 0\\ 0 & 0.54\end{bmatrix},\ \boldsymbol{W}_{21}=\begin{bmatrix}0.1 & 0\\ 0 & 0.01\end{bmatrix},\ \boldsymbol{W}_{22}=\begin{bmatrix}0.16 & 0\\ 0 & 0.01\end{bmatrix}, \tag{18a}$$

$$\boldsymbol{W}_{d_{11}}=\begin{bmatrix}0.1 & 0.01\\ 0.11 & 0.05\end{bmatrix},\ \boldsymbol{W}_{d_{12}}=\begin{bmatrix}0.03 & 0\\ 0.1 & -0.12\end{bmatrix},\ \boldsymbol{W}_{d_{21}}=\begin{bmatrix}0.02 & 0.06\\ 0 & 0.01\end{bmatrix},\ \boldsymbol{W}_{d_{22}}=\begin{bmatrix}0.15 & 0\\ 0 & 0.01\end{bmatrix}, \tag{18b}$$

$$\boldsymbol{U}_0=\boldsymbol{U}_1=\begin{bmatrix}0\\ 0.1\\ 0.1\\ 0.1\end{bmatrix},\ \boldsymbol{V}_0=\begin{bmatrix}0.01 & 0 & 0 & 0\end{bmatrix},\ \boldsymbol{V}_1=\begin{bmatrix}0 & 0.01 & 0 & 0\end{bmatrix}, \tag{18c}$$

$$d_1^v=2,\ d_2^v=9,\ k_o=0,\ k_q=1. \tag{18d}$$

The class of nonlinearities considered in this example covers zeroing, MT, saturation, MT and zeroing combinations, MT and saturation combinations, etc. The values of d_2^h (which guarantees the GAS of this system) obtained via Theorem 1 for some given values of d_1^h are shown in Table 2. It is clear from Table 2 that Theorem 1 provides better results as compared to Theorem 1 [18].

Table 2 Upper delay bound d_2^h for various d_1^h in Example 2

Methods/ d_1^h for $2 \le d^v(\nu) \le 9$	2	4	6	8
Theorem 1 [18]	16	18	20	22
Theorem 1 (Proposed)	21	23	25	27

Following [49], the GAS of the current DS is implied by

$$\begin{bmatrix}\wp_1^h(\mu+1,\ \nu)\\ \wp_2^h(\mu+1,\ \nu)\\ \wp_1^v(\mu,\ \nu+1)\\ \wp_2^v(\mu,\ \nu+1)\end{bmatrix}=\left\{\begin{bmatrix}0.54K_1 & -0.29K_1 & -0.1K_1 & 0\\ 0.2K_2 & 0.25K_2 & 0 & 0.54K_2\\ 0.1K_3 & 0 & 0.16K_3 & 0\\ 0 & 0.01K_4 & 0 & 0.01K_4\end{bmatrix}+\begin{bmatrix}0 & 0 & 0 & 0\\ 0.001\Im_0K_2 & 0 & 0 & 0\\ 0.001\Im_0K_3 & 0 & 0 & 0\\ 0.001\Im_0K_4 & 0 & 0 & 0\end{bmatrix}\right\}\begin{bmatrix}\wp_1^h(\mu,\ \nu)\\ \wp_2^h(\mu,\ \nu)\\ \wp_1^v(\mu,\ \nu)\\ \wp_2^v(\mu,\ \nu)\end{bmatrix}$$
$$+\left\{\begin{bmatrix}0.1K_1 & 0.01K_1 & 0.03K_1 & 0\\ 0.11K_2 & 0.05K_2 & 0.1K_2 & -0.12K_2\\ 0.02K_3 & 0.06K_3 & 0.15K_3 & 0\\ 0 & 0.01K_4 & 0 & 0.01K_4\end{bmatrix}+\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0.001\Im_1K_2 & 0 & 0\\ 0 & 0.001\Im_1K_3 & 0 & 0\\ 0 & 0.001\Im_1K_4 & 0 & 0\end{bmatrix}\right\}\begin{bmatrix}\wp_1^h(\mu-d^h(\mu),\ \nu)\\ \wp_2^h(\mu-d^h(\mu),\ \nu)\\ \wp_1^v(\mu,\ \nu-d^v(\nu))\\ \wp_2^v(\mu,\ \nu-d^v(\nu))\end{bmatrix}, \tag{19}$$

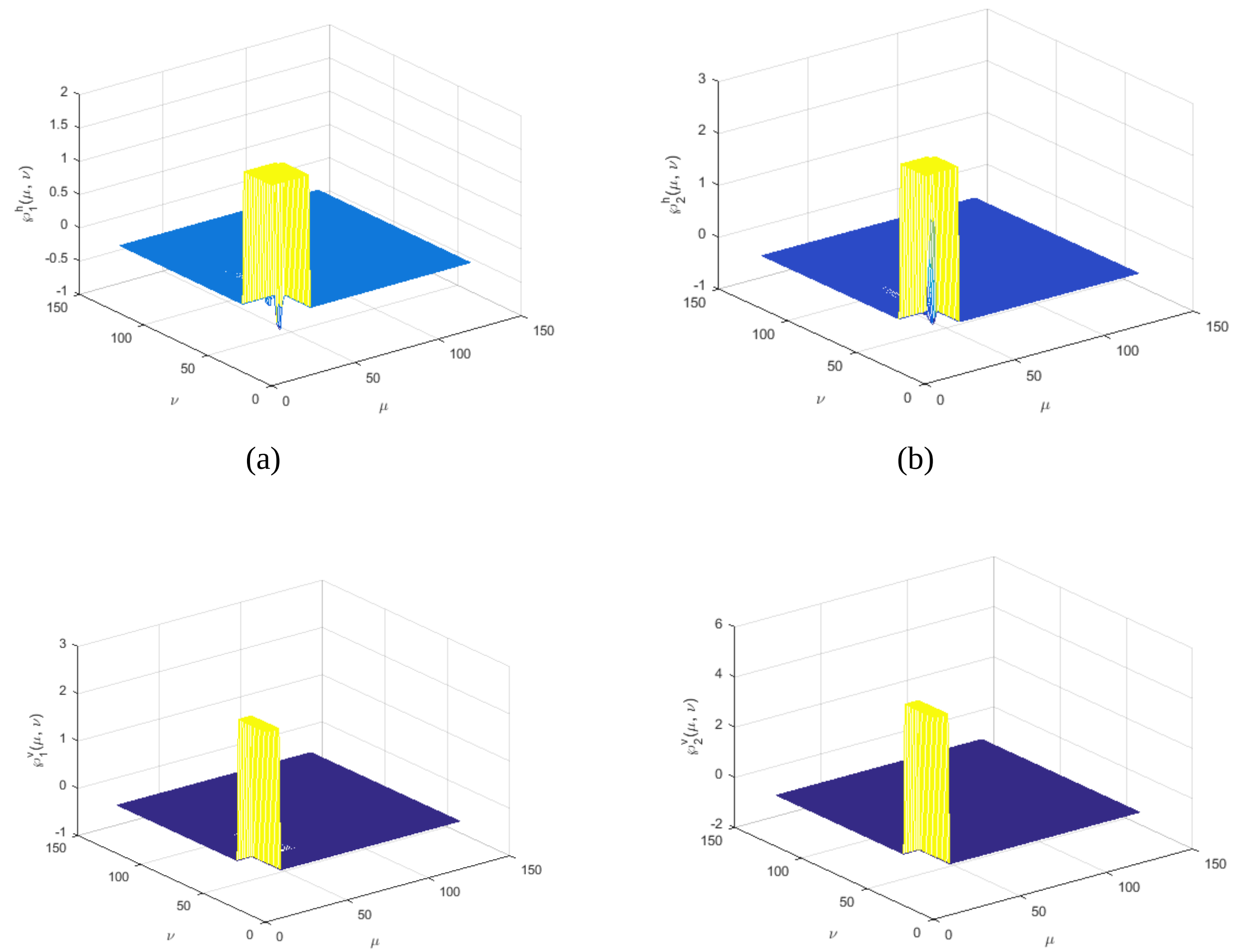
where $K_i \in [0,\ 1],\ i=1,\ 2,\ 3,\ 4.$

With $K_1 = K_2 = K_3 = K_4 = 0.2$, $\mathfrak{I}_0 = \mathfrak{I}_1 = 1$, $d^h(\mu) = \lfloor |\,20\sin(\frac{180}{\pi}(\mu-1))\,| \rceil + 2$, $d^v(\nu) = \lfloor |\,7\sin(\frac{180}{\pi}(\nu-1))\,| \rceil + 2$ and selecting the initial conditions as

$$\wp^h(\mu,\ \nu) = \begin{bmatrix} \wp_1^h(\mu,\ \nu) & \wp_2^h(\mu,\ \nu) \end{bmatrix}^{\mathrm{T}} = \begin{cases} \begin{bmatrix} 2 & 4 \end{bmatrix}^{\mathrm{T}}, & \forall\ 23 > \nu \geq 0, \quad 22 \geq \mu \geq 0, \\ \mathbf{0}, & \forall\ 23 \leq \nu, \quad 22 \geq \mu \geq 0, \end{cases} \tag{20a}$$

$$\wp^v(\mu,\ \nu) = \begin{bmatrix} \wp_1^v(\mu,\ \nu) & \wp_2^v(\mu,\ \nu) \end{bmatrix}^{\mathrm{T}} = \begin{cases} \begin{bmatrix} 2 & 4 \end{bmatrix}^{\mathrm{T}}, & \forall\ 23 > \mu \geq 0, \quad 9 \geq \nu \geq 0, \\ \mathbf{0}, & \forall\ 23 \leq \mu, \quad 9 \geq \nu \geq 0, \end{cases} \tag{20b}$$

the state trajectories of the DS under consideration are shown in Figure 4. Figure 4 shows that the system state vector tends to zero as $\mu + \nu \to \infty$. This is consistent with the GAS of the present DS.



(c) (d)

Figure 4. State trajectories for Example 2.

In the above examples, we have considered two different cases. In Example 1, we have $k_o = -1,\ k_q = 1$, which includes zeroing, saturation, MT, 2's complement overflow, triangular, MT and saturation combinations, MT and zeroing combinations, etc. In Example 2, these parameters are considered as $k_o = 0,\ k_q = 1$, which covers zeroing, MT, saturation, MT and zeroing combinations, MT and saturation combinations, etc. It is clear from Tables 1 and 2 that the proposed theorem yields improved results over [18] for these cases. Observe that, these examples fall outside the application scope of [16].

6. DISCUSSION

The key finding in this study is the delay-dependent criterion (Theorem 1) for the 2-*D* system given by (1)-(5). A flow chart for the suggested technique is shown in Figure 1.

In many situations, it may happen that the GAS of the 2-*D* system is confirmed for a particular set of k_o and k_q values, but the system displays unstable behaviour for some other set of k_o and k_q values. The set of values of k_o and k_q for which the GAS of a given 2-*D* system is assured can be determined by Theorem 1.

The matrices $\boldsymbol{H}^h$ and $\boldsymbol{H}^v$ help to lessen the conservatism of Theorem 1. The choice of these matrices as the diagonal one aids in minimizing the computational burden of Theorem 1.

The proof of Theorem 1 shows that, unlike (33), the conditions in (12) are independent of the unknown matrices $\mathfrak{I}_i (i = 0,\ 1)$. Lemma 3 has been used to eliminate $\mathfrak{I}_i (i = 0,\ 1)$ in (33) and to obtain its equivalent form (12). The equivalence of (9a) and (9b) in Lemma 3 also holds for uncertain matrix $\mathfrak{I} = \mathfrak{I}\ (\mu,\ \nu)$ satisfying $\mathfrak{I}^T(\mu,\ \nu)\mathfrak{I}(\mu,\ \nu) \le \boldsymbol{I}$. Therefore, Theorem 1 can also be used to evaluate the GAS of the system shown in (1)-(5) with $\Delta\boldsymbol{W} = \boldsymbol{U}_0\mathfrak{I}_0(\mu,\ \nu)\boldsymbol{V}_0$ and $\Delta\boldsymbol{W}_d = \boldsymbol{U}_1\mathfrak{I}_1(\mu,\ \nu)\boldsymbol{V}_1$ subject to $\mathfrak{I}_i^T(\mu,\ \nu)\mathfrak{I}_i(\mu,\ \nu) \le \boldsymbol{I}$.

The feasibility test of the conditions in Theorem 1 can be performed using the MATLAB LMI solver [29] with YALMIP 3.0 [46].

The work in [16] is primarily focussed on deriving GAS criteria for delayed DSs with saturation overflow arithmetic while ignoring any quantization effects. In contrast, the results obtained in this paper are suitable for determining the GAS of DSs operating under the influence of both quantization and overflow.

It should be noted that the method used in this study utilizes a constant Lyapunov function, which may produce conservative stability results. However, by using parameter-dependent Lyapunov functions together with more accurate characterization of nonlinearities, uncertainties and delays, the presented results can be improved further. The obtained results provide only sufficient conditions. More research is needed to close the gap between ‘sufficiency’ and ‘necessity’ for the GAS of a 2-*D* system, which occurs in the current approach.

7. CONCLUSION

A new delay-dependent criterion for testing the GAS of 2-*D* uncertain DSs with TVDs and FWNs has been presented. The approach is quite distinct and leads to improved GAS results than [18].

The potential application of the presented method to analyze the stability for 2-*D* DSs with TVDs, FWNs and external interference appears to be an interesting problem to investigate further. The concepts discussed in this work can be extended to a class of delayed systems using polytopic uncertainties [50] and FWNs, which need further exploration. The obtained results can be easily extended to ϱ-dimensional (ϱ > 2) systems.

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APPENDIX I

Proof of Theorem 1: Assume that (21a), (21b) and (22) provide the terms $\boldsymbol{\kappa}^h(\mu,\ \nu)$, $\boldsymbol{\kappa}^v(\mu,\ \nu)$ and $\boldsymbol{\varsigma}(\mu,\ \nu)$, respectively.

$$\boldsymbol{\kappa}^h(\mu,\ \nu)=\wp^h(\mu+1,\ \nu)-\wp^h(\mu,\ \nu)=\boldsymbol{f}^h(\sigma^h(\mu,\ \nu))-\wp^h(\mu,\ \nu), \tag{21a}$$

$$\boldsymbol{\kappa}^v(\mu,\ \nu)=\wp^v(\mu,\ \nu+1)-\wp^v(\mu,\ \nu)=\boldsymbol{f}^v(\sigma^v(\mu,\ \nu))-\wp^v(\mu,\ \nu), \tag{21b}$$

$$\boldsymbol{\varsigma}(\mu,\ \nu)=col\{\wp(\mu,\ \nu),\ \wp_1(\mu,\ \nu),\ \wp_2(\mu,\ \nu),\ \wp_3(\mu,\ \nu),\ \wp_4(\mu,\ \nu),\ \wp_5(\mu,\ \nu),\ \wp_6(\mu,\ \nu),\ \boldsymbol{f}(\sigma(\mu,\ \nu))\}, \tag{22}$$

where

$$\wp(\mu,\ \nu)=\begin{bmatrix}\wp^h(\mu,\ \nu)\\ \hdashline \wp^v(\mu,\ \nu)\end{bmatrix},\ \wp_1(\mu,\ \nu)=\begin{bmatrix}\wp^h(\mu-d^h(\mu),\ \nu)\\ \hdashline \wp^v(\mu,\ \nu-d^v(\nu))\end{bmatrix},\ \wp_2(\mu,\ \nu)=\begin{bmatrix}\wp^h(\mu-d_1^h,\ \nu)\\ \hdashline \wp^v(\mu,\ \nu-d_1^v)\end{bmatrix},\ \wp_3(\mu,\ \nu)=\begin{bmatrix}\wp^h(\mu-d_2^h,\ \nu)\\ \hdashline \wp^v(\mu,\ \nu-d_2^v)\end{bmatrix},$$

$$\wp_4(\mu,\ \nu)=\begin{bmatrix}\boldsymbol{\chi}(\mu,\ 0,\ d_1^h)\\ \hdashline \boldsymbol{\chi}(\nu,\ 0,\ d_1^v)\end{bmatrix},\ \wp_5(\mu,\ \nu)=\begin{bmatrix}\boldsymbol{\chi}(\mu,\ d_1^h,\ d^h(\mu))\\ \hdashline \boldsymbol{\chi}(\nu,\ d_1^v,\ d^v(\nu))\end{bmatrix},\ \wp_6(\mu,\ \nu)=\begin{bmatrix}\boldsymbol{\chi}(\mu,\ d^h(\mu),\ d_2^h)\\ \hdashline \boldsymbol{\chi}(\nu,\ d^v(\nu),\ d_2^v)\end{bmatrix}.$$

Equations (23)-(25) represent a 2-*D* quadratic Lyapunov functional that is taken into account:

$$V(\wp(\mu,\ \nu))=\sum_{i=1}^{3}V_i(\wp(\mu,\ \nu)) \tag{23}$$

$$V_1(\wp(\mu,\ \nu))=\boldsymbol{\xi}^{h^{\mathrm{T}}}(\mu,\ \nu)\boldsymbol{O}^h\boldsymbol{\xi}^h(\mu,\ \nu)+\boldsymbol{\xi}^{v^{\mathrm{T}}}(\mu,\ \nu)\boldsymbol{O}^v\boldsymbol{\xi}^v(\mu,\ \nu), \tag{24a}$$

$$
\begin{aligned}
V_2(\wp(\mu,\ \nu)) = {} & \sum_{r=\mu-d_1^h}^{\mu-1} \wp^{h^T}(r,\ \nu)\boldsymbol{E}_1^h \wp^h(r,\ \nu) + \sum_{r=\mu-d_2^h}^{\mu-d_1^h-1} \wp^{h^T}(r,\ \nu)\boldsymbol{E}_2^h \wp^h(r,\ \nu) \\
& + \sum_{s=-d_2^h}^{d_1^h} \sum_{r=\mu+s}^{\mu-1} \wp^{\mathrm{T}}(r,\ \nu)\boldsymbol{E}_3^h \wp(r,\ \nu) + \sum_{r=\nu-d_1^v}^{\nu-1} \wp^{v^{\mathrm{T}}}(\mu,\ r)\boldsymbol{E}_1^v \wp^v(\mu,\ r) \\
& + \sum_{r=\nu-d_2^v}^{\nu-d_1^v-1} \wp^{v^{\mathrm{T}}}(\mu,\ r)\boldsymbol{E}_2^v \wp^v(\mu,\ r) + \sum_{s=-d_2^v}^{d_1^v} \sum_{r=\nu+s}^{\nu-1} \wp^{\mathrm{T}}(\mu,\ r)\boldsymbol{E}_3^v \wp(\mu,\ r), \qquad (24b)
\end{aligned}
$$

$$
\begin{aligned}
V_3(\wp(\mu,\ \nu)) = {} & d_1^h \sum_{s=-d_1^h}^{-1} \sum_{r=\mu+s}^{\mu-1} \boldsymbol{\kappa}^{h^{\mathrm{T}}}(r,\ \nu)\boldsymbol{T}_1^h \boldsymbol{\kappa}^h(r,\ \nu) + d_{12}^h \sum_{s=-d_2^h}^{-d_1^h-1} \sum_{r=\mu+s}^{\mu-1} \boldsymbol{\kappa}^{h^{\mathrm{T}}}(r,\ \nu)\boldsymbol{T}_1^h \boldsymbol{\kappa}^h(r,\ \nu) \\
& + d_1^v \sum_{s=-d_1^v}^{-1} \sum_{r=\nu+s}^{\nu-1} \boldsymbol{\kappa}^{v^{\mathrm{T}}}(\mu,\ r)\boldsymbol{T}_1^v \boldsymbol{\kappa}^v(\mu,\ r) + d_{12}^v \sum_{s=-d_2^v}^{-d_1^v-1} \sum_{r=\nu+s}^{\nu-1} \boldsymbol{\kappa}^{v^{\mathrm{T}}}(\mu,\ r)\boldsymbol{T}_1^v \boldsymbol{\kappa}^v(\mu,\ r), \qquad (24c)
\end{aligned}
$$

where

$$
\boldsymbol{\xi}^h(\mu,\ \nu) = \left[\wp^{h^{\mathrm{T}}}(\mu,\ \nu) \quad \sum_{r=\mu-d_1^h}^{\mu-1} \wp^{h^{\mathrm{T}}}(r,\ \nu) \quad \sum_{r=\mu-d_2^h}^{\mu-d_1^h-1} \wp^{h^{\mathrm{T}}}(r,\ \nu) \right]^{\mathrm{T}}, \qquad (25a)
$$

$$
\boldsymbol{\xi}^v(\mu,\ \nu) = \left[\wp^{v^{\mathrm{T}}}(\mu,\ \nu) \quad \sum_{r=\nu-d_1^v}^{\nu-1} \wp^{v^{\mathrm{T}}}(\mu,\ r) \quad \sum_{r=\nu-d_2^v}^{\nu-d_1^v-1} \wp^{v^{\mathrm{T}}}(\mu,\ r) \right]^{\mathrm{T}}. \qquad (25b)
$$

The above 2-*D* Lyapunov functional is an extension of 1-*D* Lyapunov functional used in [48]. Equations (26) and (27) give the forward difference of (23) along the trajectories of the system.

$$
\Delta V(\wp(\mu,\ \nu)) = \sum_{i=1}^{3} \Delta V_i(\wp(\mu,\ \nu)), \qquad (26)
$$

where

$$
\begin{aligned}
\Delta V_1(\wp(\mu,\ \nu)) = {} & \boldsymbol{\xi}^{h^{\mathrm{T}}}(\mu+1,\ \nu)\boldsymbol{O}^h \boldsymbol{\xi}^h(\mu+1,\ \nu) - \boldsymbol{\xi}^{h^{\mathrm{T}}}(\mu,\ \nu)\boldsymbol{O}^h \boldsymbol{\xi}^h(\mu,\ \nu) \\
& + \boldsymbol{\xi}^{v^{\mathrm{T}}}(\mu,\ \nu+1)\boldsymbol{O}^v \boldsymbol{\xi}^v(\mu,\ \nu+1) - \boldsymbol{\xi}^{v^{\mathrm{T}}}(\mu,\ \nu)\boldsymbol{O}^v \boldsymbol{\xi}^v(\mu,\ \nu) \\
= {} & \boldsymbol{\varsigma}^T(\mu,\ \nu)\bar{\boldsymbol{\Phi}}(d^h(\mu),\ d^v(\nu))\boldsymbol{\varsigma}(\mu,\ \nu), \qquad (27a)
\end{aligned}
$$

$$
\bar{\boldsymbol{\Phi}}(d^h(\mu),\ d^v(\nu)) =
$$

$$
\begin{bmatrix}
-\boldsymbol{O}+(\boldsymbol{O}_2+\boldsymbol{O}_2^T)/2 & \boldsymbol{0} & (\boldsymbol{O}_3-\boldsymbol{O}_2)/2 & -\boldsymbol{O}_3/2 & \Phi_{15} & \Phi_{16} & \Phi_{17} & \boldsymbol{O}_2^T/2 \\
* & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
* & * & \boldsymbol{0} & \boldsymbol{0} & \Phi_{35} & \Phi_{36} & \boldsymbol{Y}_3(\boldsymbol{O}_6-\boldsymbol{O}_5)/2 & (\boldsymbol{O}_3^T-\boldsymbol{O}_2^T)/2 \\
* & * & * & \boldsymbol{0} & -\boldsymbol{Y}_1\boldsymbol{O}_5^T/2 & -\boldsymbol{Y}_2\boldsymbol{O}_6/2 & -\boldsymbol{Y}_3\boldsymbol{O}_6/2 & -\boldsymbol{O}_3^T/2 \\
* & * & * & * & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{Y}_1\boldsymbol{O}_2^T/2 \\
* & * & * & * & * & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{Y}_2\boldsymbol{O}_3^T/2 \\
* & * & * & * & * & * & \boldsymbol{0} & \boldsymbol{Y}_3\boldsymbol{O}_3^T/2 \\
* & * & * & * & * & * & * & \boldsymbol{O}_1
\end{bmatrix}, \qquad (27b)
$$

$$
\begin{aligned}
\Delta V_2(\wp(\mu,\ \nu)) = {} & \wp^{h^{\mathrm{T}}}(\mu,\ \nu)\boldsymbol{E}_1^h\wp^h(\mu,\ \nu) + \wp^{\nu^{\mathrm{T}}}(\mu,\ \nu)\boldsymbol{E}_1^\nu\wp^\nu(\mu,\ \nu) - \wp^{h^{\mathrm{T}}}(\mu - d_1^h,\ \nu)\boldsymbol{E}_1^h\wp^h(\mu - d_1^h,\ \nu) \\
& - \wp^{\nu^{\mathrm{T}}}(\mu,\ \nu - d_1^\nu)\boldsymbol{E}_1^\nu\wp^\nu(\mu,\ \nu - d_1^\nu) + \wp^{h^{\mathrm{T}}}(\mu - d_1^h,\ \nu)\boldsymbol{E}_2^h\wp^h(\mu - d_1^h,\ \nu) \\
& - \wp^{\nu^{T}}(\mu,\ \nu - d_2^\nu)\boldsymbol{E}_2^\nu\wp^\nu(\mu,\ \nu - d_2^\nu) - \wp^{h^{\mathrm{T}}}(\mu - d_2^h,\ \nu)\boldsymbol{E}_2^h\wp^h(\mu - d_2^h,\ \nu) \\
& + \wp^{\nu^{\mathrm{T}}}(\mu,\ \nu - d_1^\nu)\boldsymbol{E}_2^\nu\wp^\nu(\mu,\ \nu - d_1^\nu) - \sum_{r=\mu-d_2^h}^{\mu-d_1^h}\wp^T(r,\ \nu)\boldsymbol{E}_3^h\wp(r,\ \nu) - \sum_{r=\nu-d_2^\nu}^{\nu-d_1^\nu}\wp^T(\mu,\ r)\boldsymbol{E}_3^\nu\wp(\mu,\ r),
\end{aligned}
\tag{27c}
$$

$$
\begin{aligned}
\Delta V_3(\wp(\mu,\ \nu)) = {} & \boldsymbol{\kappa}^{h^{\mathrm{T}}}(\mu,\ \nu)(d_1^{h^2}\boldsymbol{T}_1^h)\boldsymbol{\kappa}^h(\mu,\ \nu) + \boldsymbol{\kappa}^{h^{\mathrm{T}}}(\mu,\ \nu)(d_{12}^{h^2}\boldsymbol{T}_2^h)\boldsymbol{\kappa}^h(\mu,\ \nu) + \boldsymbol{\kappa}^{\nu^{\mathrm{T}}}(\mu,\ \nu)(d_1^{\nu^2}\boldsymbol{T}_1^\nu)\boldsymbol{\kappa}^\nu(\mu,\ \nu) \\
& + \boldsymbol{\kappa}^{\nu^{\mathrm{T}}}(\mu,\ \nu)(d_{12}^{\nu^2}\boldsymbol{T}_2^\nu)\boldsymbol{\kappa}^\nu(\mu,\ \nu) + \sum_{i=1}^{2}(\boldsymbol{S}_i^h(\mu,\ \nu) + \boldsymbol{S}_i^\nu(\mu,\ \nu)),
\end{aligned}
\tag{27d}
$$

$$
\boldsymbol{S}_1^h(\mu,\ \nu) = -d_1^h\sum_{s=\mu-d_1^h}^{\mu-1}\boldsymbol{\kappa}^{h^T}(s,\ \nu)\boldsymbol{T}_1^h\boldsymbol{\kappa}^h(s,\ \nu), \tag{27e}
$$

$$
\boldsymbol{S}_1^\nu(\mu,\ \nu) = -d_1^\nu\sum_{s=\nu-d_1^\nu}^{\nu-1}\boldsymbol{\kappa}^{\nu^T}(\mu,\ s)\boldsymbol{T}_1^\nu\boldsymbol{\kappa}^\nu(\mu,\ s), \tag{27f}
$$

$$
\boldsymbol{S}_2^h(\mu,\ \nu) = -d_{12}^h\sum_{s=\mu-d_2^h}^{\mu-d_1^h-1}\boldsymbol{\kappa}^{h^T}(s,\ \nu)\boldsymbol{T}_2^h\boldsymbol{\kappa}^h(s,\ \nu), \tag{27g}
$$

$$
\boldsymbol{S}_2^\nu(\mu,\ \nu) = -d_{12}^\nu\sum_{s=\nu-d_2^\nu}^{\nu-d_1^\nu-1}\boldsymbol{\kappa}^{\nu^T}(\mu,\ s)\boldsymbol{T}_2^\nu\boldsymbol{\kappa}^\nu(\mu,\ s). \tag{27h}
$$

Using Lemmas 1, 2 and following [47], it is easy to show that

$$
\Delta V(\wp(\mu,\ \nu)) \leqslant \boldsymbol{\varsigma}^T(\mu,\ \nu)\boldsymbol{\Psi}(d^h(\mu),\ d^\nu(\nu))\boldsymbol{\varsigma}(\mu,\ \nu) - 2\delta, \tag{28}
$$

where

$$
\delta = [k_q\boldsymbol{\sigma}(\mu,\ \nu) - \boldsymbol{f}(\boldsymbol{\sigma}(\mu,\ \nu))]^T\boldsymbol{G}[\boldsymbol{f}(\boldsymbol{\sigma}(\mu,\ \nu)) - k_o\boldsymbol{\sigma}(\mu,\ \nu)], \tag{29}
$$

$$
\boldsymbol{\Psi}(d^h(\mu),\ d^\nu(\nu)) =
$$

$$
\begin{bmatrix}
\Phi_{11} - 2k_qk_o\bar{\boldsymbol{W}}^T\boldsymbol{G}\bar{\boldsymbol{W}} & -2k_qk_o\bar{\boldsymbol{W}}^T\boldsymbol{G}\bar{\boldsymbol{W}}_d & (\boldsymbol{O}_3 - \boldsymbol{O}_2)/2 - 2\boldsymbol{T}_1 & -\boldsymbol{O}_3/2 & \Phi_{15} + 3\boldsymbol{T}_1 & \Phi_{16} & \Phi_{17} & \Phi_{18} + \boldsymbol{O}_2^T/2 + (k_q + k_o)\bar{\boldsymbol{W}}^T\boldsymbol{G} \\
* & \Phi_{22} - 2k_qk_o\bar{\boldsymbol{W}}_d^T\boldsymbol{G}\bar{\boldsymbol{W}}_d & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & (k_q + k_o)\bar{\boldsymbol{W}}_d^T\boldsymbol{G} \\
* & * & \boldsymbol{0} & \boldsymbol{0} & \Phi_{35} & \Phi_{36} & \Phi_{37} & (\boldsymbol{O}_3^T - \boldsymbol{O}_2^T)/2 \\
* & * & * & \boldsymbol{0} & -\boldsymbol{Y}_1\boldsymbol{O}_5^T/2 & \Phi_{46} & \Phi_{47} & -\boldsymbol{O}_3^T/2 \\
* & * & * & * & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{Y}_1\boldsymbol{O}_2^T/2 \\
* & * & * & * & * & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{Y}_2\boldsymbol{O}_3^T/2 \\
* & * & * & * & * & * & \boldsymbol{0} & \boldsymbol{Y}_3\boldsymbol{O}_3^T/2 \\
* & * & * & * & * & * & * & \boldsymbol{O}_1 - \Phi_{18} - 2\boldsymbol{G}
\end{bmatrix}, \tag{30}
$$

$$
\bar{\boldsymbol{W}} = \boldsymbol{W} + \Delta\boldsymbol{W},\ \bar{\boldsymbol{W}}_d = \boldsymbol{W}_d + \Delta\boldsymbol{W}_d. \tag{31}
$$

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In view of (3), δ given by (29), is non-negative [31]. Therefore, the condition $\boldsymbol{\Psi}(d^h(\mu),\ d^v(\nu)) < \mathbf{0}$ together with (10) and (11) leads to $\Delta V(\wp(\mu,\ \nu)) < 0$.

Now, following [8], one can show that $\lim\limits_{\mu\to\infty \text{ and/or } \nu\to\infty} \wp(\mu,\ \nu) = \lim\limits_{\mu+\nu\to\infty} \wp(\mu,\ \nu) = \mathbf{0}$ for the boundary conditions given by (5). This confirms the GAS of the system under consideration.

Next, the condition $\boldsymbol{\Psi}(d^h(\mu),\ d^v(\nu)) < \mathbf{0}$ can be rearranged as

$$\begin{bmatrix} \Phi_{11} & \mathbf{0} & (\boldsymbol{O}_3-\boldsymbol{O}_2)/2-2\boldsymbol{T}_1 & -\boldsymbol{O}_3/2 & \Phi_{15}+3\boldsymbol{T}_1 & \Phi_{16} & \Phi_{17} & \Phi_{18}+\boldsymbol{O}_2^T/2+k_q\bar{\boldsymbol{W}}^T\boldsymbol{G} \\ * & \Phi_{22} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & k_q\bar{\boldsymbol{W}}_d^T\boldsymbol{G} \\ * & * & \mathbf{0} & \mathbf{0} & \Phi_{35} & \Phi_{36} & \Phi_{37} & (\boldsymbol{O}_3^T-\boldsymbol{O}_2^T)/2 \\ * & * & * & \mathbf{0} & -\boldsymbol{Y}_1\boldsymbol{O}_5^T/2 & \Phi_{46} & \Phi_{47} & -\boldsymbol{O}_3^T/2 \\ * & * & * & * & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{Y}_1\boldsymbol{O}_2^T/2 \\ * & * & * & * & * & \mathbf{0} & \mathbf{0} & \boldsymbol{Y}_2\boldsymbol{O}_3^T/2 \\ * & * & * & * & * & * & \mathbf{0} & \boldsymbol{Y}_3\boldsymbol{O}_3^T/2 \\ * & * & * & * & * & * & * & \Phi_{88} \end{bmatrix} - \begin{bmatrix} -k_q\sqrt{-2k_o}\bar{\boldsymbol{W}}^T\boldsymbol{G} \\ -k_q\sqrt{-2k_o}\bar{\boldsymbol{W}}_d^T\boldsymbol{G} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \sqrt{\frac{-k_o}{2}}\boldsymbol{G} \end{bmatrix}$$

$$\times\left[-k_q\boldsymbol{G}\right]^{-1}\times\left[-k_q\sqrt{-2k_o}\boldsymbol{G}\bar{\boldsymbol{W}} \quad -k_q\sqrt{-2k_o}\boldsymbol{G}\bar{\boldsymbol{W}}_d \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \sqrt{\frac{-k_o}{2}}\boldsymbol{G}\right] < \mathbf{0}. \tag{32}$$

By Schur's complement, (32) is equivalent to

$$\begin{bmatrix} \Phi_{11} & \mathbf{0} & (\boldsymbol{O}_3-\boldsymbol{O}_2)/2-2T_1 & -\boldsymbol{O}_3/2 & \Phi_{15}+3T_1 & \Phi_{16} & \Phi_{17} & \Phi_{18}+\boldsymbol{O}_2^T/2+k_q\bar{\boldsymbol{W}}^T\boldsymbol{G} & -k_q\sqrt{-2k_o}\bar{\boldsymbol{W}}^T\boldsymbol{G} \\ * & \Phi_{22} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & k_q\bar{\boldsymbol{W}}_d^T\boldsymbol{G} & -k_q\sqrt{-2k_o}\bar{\boldsymbol{W}}_d^T\boldsymbol{G} \\ * & * & \mathbf{0} & \mathbf{0} & \Phi_{35} & \Phi_{36} & \Phi_{37} & (\boldsymbol{O}_3^T-\boldsymbol{O}_2^T)/2 & \mathbf{0} \\ * & * & * & \mathbf{0} & -\mathrm{Y}_1\boldsymbol{O}_5^T/2 & \Phi_{46} & \Phi_{47} & -\boldsymbol{O}_3^T/2 & \mathbf{0} \\ * & * & * & * & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{Y}_1\boldsymbol{O}_2^T/2 & \mathbf{0} \\ * & * & * & * & * & \mathbf{0} & \mathbf{0} & \boldsymbol{Y}_2\boldsymbol{O}_3^T/2 & \mathbf{0} \\ * & * & * & * & * & * & \mathbf{0} & \boldsymbol{Y}_3\boldsymbol{O}_3^T/2 & \mathbf{0} \\ * & * & * & * & * & * & * & \Phi_{88} & \sqrt{\frac{-k_o}{2}}\boldsymbol{G} \\ * & * & * & * & * & * & * & * & -k_q\boldsymbol{G} \end{bmatrix} < \mathbf{0}. \tag{33}$$

Employing (4) and Lemma 3, one can prove that $\boldsymbol{\Psi}(d^h(\mu),\ d^v(\nu)) < \mathbf{0}$ is equivalent to $\boldsymbol{\Phi}(d^h(\mu),\ d^v(\nu)) < \mathbf{0}$. It is obvious that $\boldsymbol{\Phi}(d^h(\mu),\ d^v(\nu))$ is an affine matrix function with respect to $d^h(\mu)$ and $d^v(\nu)$. Thus, the condition $\boldsymbol{\Phi}(d^h(\mu),\ d^v(\nu)) < \mathbf{0}$ is fulfilled if (12) holds. This completes the proof.

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