2x2 Matrix: Calculation of The exponential

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November 26, 2025

Abstract

This work explores a robust method for calculating the exponential of a 2x2 matrix without resorting to diagonalization. This approach significantly simplifies complex matrix operations.

We first present our previously known and published result, which is a single formula for calculating the nth power of any 2x2 matrix, whether it is diagonalizable or not, whether it has two distinct eigenvalues or not, and whether these eigenvalues are real or not.

Finally, as we did for the powers of a 2x2 matrix, we announce and prove our new formula, which gives the exponential of any 2x2 matrix. That is to say, a single formula for calculating the exponential of any 2x2 matrix, whether it is diagonalizable or not, whether it has two distinct eigenvalues or not, and whether these eigenvalues are real or not.

Keywords: Power of a matrix; Exponential of a matrix; Eigenvalues.

1 Introduction:

In our two previous articles [1] and [2], which deal with 2x2 matrices, we obtained a new formula for the nth powers of this type of matrix.

$$A^n = \begin{pmatrix} a\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} - \sum_{k=0}^{n-2}\alpha_1^{k+1}\alpha_2^{n-1-k} & b\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} \\ c\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} & \sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k} - a\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} \end{pmatrix}$$

This original formula was more general than those previously announced, such as those by Kenneth S. Williams (see [3]) in 1992

$$A^n = \begin{cases} \alpha^n\left(\frac{A-\beta I}{\alpha-\beta}\right) + \beta^n\left(\frac{A-\alpha I}{\beta-\alpha}\right) & \text{if } \alpha \neq \beta \\ \alpha^{n-1}\left(nA - (n-1)\alpha I\right) & \text{if } \alpha = \beta \end{cases}$$

or even in the article [4]. This novelty was summarized in a single formula for the nth power of any 2x2 matrix, whether diagonalizable or not, whether it has two distinct eigenvalues or not, and whether these eigenvalues are real or not.

Furthermore, in the article [8], the authors seek to determine other combinatorial identities derived from the nth power of a 2x2 matrix. But the real problem for mathematicians, and even computer scientists, is determining the exponential of a matrix (see [5], [6], [7], [12]), as this problem frequently arises, particularly in systems of differential equations of the type $X(t)' = AX(t)$ (see [9], [10], [11]).

Here, we present, for the first time, a single formula (for all uses) for the exponential of any 2x2 matrix.

These results can be applied to fields such as recurrence relations, differential equations, dynamical systems, and computer algorithms, where powers of small matrices frequently appear.

2 The power of any (diagonalizable or non-diagonalizable) matrix of order 2:

Theorem 1 *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *a 2X2 matrix, so* $\forall n \geq 2$ $(n \in IN)$

$$A^n = \begin{pmatrix} a\sum\limits_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - \sum\limits_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k} & b\sum\limits_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} \\ c\sum\limits_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} & \sum\limits_{k=0}^{n} \alpha_1^k \alpha_2^{n-k} - a\sum\limits_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} \end{pmatrix}$$

with α_1 *and* α_2 *solutions of* $|A - \alpha I| = 0$.

Proof. We will demonstrate our formula using recurrence.

(i) For $n = 2$, we will show that we actually have

$$A^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix}$$

$$A^2 = \begin{pmatrix} a\sum_{k=0}^{1} \alpha_1^k \alpha_2^{1-k} - \alpha_1\alpha_2 & b\sum_{k=0}^{1} \alpha_1^k \alpha_2^{1-k} \\ c\sum_{k=0}^{1} \alpha_1^k \alpha_2^{1-k} & \sum_{k=0}^{2} \alpha_1^k \alpha_2^{2-k} - a\sum_{k=0}^{1} \alpha_1^k \alpha_2^{1-k} \end{pmatrix}$$

we must therefore show the following 4 equalities

$$\begin{cases} a^2 + bc = a\sum_{k=0}^{1} \alpha_1^k \alpha_2^{1-k} - \alpha_1\alpha_2 \\ d^2 + bc = \sum_{k=0}^{2} \alpha_1^k \alpha_2^{2-k} - a\sum_{k=0}^{1} \alpha_1^k \alpha_2^{1-k} \\ b(a+d) = b\sum_{k=0}^{1} \alpha_1^k \alpha_2^{1-k} \\ c(a+d) = c\sum_{k=0}^{1} \alpha_1^k \alpha_2^{1-k} \end{cases}$$

what does it mean

$$\begin{cases} a^2 + bc = a(\alpha_2 + \alpha_1) - \alpha_1\alpha_2 \\ d^2 + bc = \alpha_2^2 + \alpha_1\alpha_2 + \alpha_1^2 - a(\alpha_2 + \alpha_1) \\ b(a+d) = b(\alpha_2 + \alpha_1) \\ c(a+d) = c(\alpha_2 + \alpha_1) \end{cases}$$

α_1 and α_2 solutions of $|A-\alpha I|=0$, so

$$\left\{\begin{array}{c}\alpha_1+\alpha_2=Tr\left(A\right)=a+d\\ \alpha_1\alpha_2=|A|=ad-bc\end{array}\right.$$

So
$a\left(\alpha_2+\alpha_1\right)-\alpha_1\alpha_2=a\left(a+d\right)-\left(ad-bc\right)=a^2+ad-ad+bc=a^2+bc$
$\alpha_2^2+\alpha_1\alpha_2+\alpha_1^2-a\left(\alpha_2+\alpha_1\right)=\left(\alpha_1+\alpha_2\right)^2-\alpha_1\alpha_2-a\left(\alpha_2+\alpha_1\right)=\left(a+d\right)^2-\left(ad-bc\right)-a\left(a+d\right)=d^2+bc$
$b\left(\alpha_2+\alpha_1\right)=b\left(a+d\right)$
$c\left(\alpha_2+\alpha_1\right)=c\left(a+d\right)$
hence the result is true for $n=2$.
(ii) Let's suppose

$$A^n=\begin{pmatrix} a\sum\limits_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}-\sum\limits_{k=0}^{n-2}\alpha_1^{k+1}\alpha_2^{n-1-k} & b\sum\limits_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}\\ c\sum\limits_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} & \sum\limits_{k=0}^{n}\alpha_1^k\alpha_2^{n-k}-a\sum\limits_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}\end{pmatrix}$$

(iii) Let us show that
$A^{n+1}=A^n.A$

$$A^{n+1}=\begin{pmatrix} a\sum\limits_{k=0}^{n}\alpha_1^k\alpha_2^{n-k}-\sum\limits_{k=0}^{n-1}\alpha_1^{k+1}\alpha_2^{n-k} & b\sum\limits_{k=0}^{n}\alpha_1^k\alpha_2^{n-k}\\ c\sum\limits_{k=0}^{n}\alpha_1^k\alpha_2^{n-k} & \sum\limits_{k=0}^{n+1}\alpha_1^k\alpha_2^{n+1-k}-a\sum\limits_{k=0}^{n}\alpha_1^k\alpha_2^{n-k}\end{pmatrix}$$

We have $A^{n+1}=A^n.A=$

$$=\begin{pmatrix} a\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}-\sum_{k=0}^{n-2}\alpha_1^{k+1}\alpha_2^{n-1-k} & b\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}\\ c\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} & \sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k}-a\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}\end{pmatrix}A$$

$$A^{n+1}=\begin{pmatrix} x & y\\ z & t\end{pmatrix}$$

$x=a^2\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}-a\sum_{k=0}^{n-2}\alpha_1^{k+1}\alpha_2^{n-1-k}+cb\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}$
$y=ba\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}-b\sum_{k=0}^{n-2}\alpha_1^{k+1}\alpha_2^{n-1-k}+db\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}$
$z=ac\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}+c\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k}-ca\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}$
$t=bc\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}+d\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k}-da\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}$

$$\Longrightarrow\left\{\begin{array}{l} x=\left(a^2+cb\right)\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}-a\sum_{k=0}^{n-2}\alpha_1^{k+1}\alpha_2^{n-1-k}\\ y=b\left[\left(a+d\right)\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}-\sum_{k=0}^{n-2}\alpha_1^{k+1}\alpha_2^{n-1-k}\right]\\ z=c\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k}\\ t=\left(bc-da\right)\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}+d\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k}\end{array}\right.$$

$\Longrightarrow$
$x=\left[a\left(\alpha_1+\alpha_2\right)-\alpha_1\alpha_2\right]\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}-a\sum_{k=0}^{n-2}\alpha_1^{k+1}\alpha_2^{n-1-k}$
$y=b\left[\left(\alpha_1+\alpha_2\right)\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}-\sum_{k=0}^{n-2}\alpha_1^{k+1}\alpha_2^{n-1-k}\right]$
$z=c\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k}$
$t=-\alpha_1\alpha_2\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}+d\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k}$
because $\alpha_1+\alpha_2=a+d$, $\alpha_1\alpha_2=ad-bc$, $a\left(\alpha_1+\alpha_2\right)-\alpha_1\alpha_2=a^2+cb$

$$A^{n+1}=\begin{pmatrix} a\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k}-\alpha_1\alpha_2\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} & b\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k}\\ c\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k} & \sum_{k=0}^{n+1}\alpha_1^k\alpha_2^{n+1-k}-a\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k}\end{pmatrix}$$

because
$x=\left[a\left(\alpha_1+\alpha_2\right)-\alpha_1\alpha_2\right]\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}-a\sum_{k=0}^{n-2}\alpha_1^{k+1}\alpha_2^{n-1-k}=$
$=a\left(\alpha_1+\alpha_2\right)\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}-\alpha_1\alpha_2\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}-a\sum_{k=0}^{n-2}\alpha_1^{k+1}\alpha_2^{n-1-k}$
$=a\alpha_1\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}+a\alpha_2\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}-\alpha_1\alpha_2\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}-a\sum_{k=0}^{n-2}\alpha_1^{k+1}\alpha_2^{n-1-k}$
$=a\alpha_1^n+a\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-k}-\alpha_1\alpha_2\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}$
$=a\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k}-\alpha_1\alpha_2\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}$;
$y=b\left[\left(\alpha_1+\alpha_2\right)\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}-\sum_{k=0}^{n-2}\alpha_1^{k+1}\alpha_2^{n-1-k}\right]$
$=b\left[\sum_{k=0}^{n-1}\alpha_1^{k+1}\alpha_2^{n-1-k}+\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-k}-\sum_{k=0}^{n-2}\alpha_1^{k+1}\alpha_2^{n-1-k}\right]$
$=b\left[\alpha_1^n+\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-k}\right]=b\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k}$
and
$t=-\alpha_1\alpha_2\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}+d\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k}$
$=-\sum_{k=0}^{n-1}\alpha_1^{k+1}\alpha_2^{n-k}+\left(\alpha_1+\alpha_2-a\right)\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k}$ (as $\alpha_1+\alpha_2=a+d$)
$=-\sum_{k=0}^{n-1}\alpha_1^{k+1}\alpha_2^{n-k}-a\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k}+\sum_{k=0}^{n}\alpha_1^{k+1}\alpha_2^{n-k}+\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n+1-k}$
$=-a\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k}+\sum_{k=0}^{n+1}\alpha_1^k\alpha_2^{n+1-k}$
hence the result. ■

Remark 2 *Let* $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *a 2X2 matrix and* α_1 *and* α_2 *solutions of* $|A-\alpha I|=0$.

1) If $\alpha_1=\alpha_2=\alpha$, *so*

$$A^n=\left(\frac{a+d}{2}\right)^{n-1}\begin{pmatrix} \frac{(n+1)a-(n-1)d}{2} & nb \\ nc & \frac{(n+1)d-(n-1)a}{2} \end{pmatrix},\ \forall n\geq 2\ (n\in IN)$$

with $\alpha=\frac{a+d}{2}$ *and* $(a+d)^2=4(ad-bc)$ *(*$(a-d)^2=-4bc$*)*
And

$$A^n=n\alpha^{n-1}A-(n-1)\alpha^n I$$

2) If $\alpha_1\neq\alpha_2$ *so* $\forall n\geq 2\ (n\in IN)$

$$\begin{aligned} A^n &= \frac{1}{\alpha_1-\alpha_2}\begin{pmatrix} (a-\alpha_2)\alpha_1^n-(a-\alpha_1)\alpha_2^n & b\left(\alpha_1^n-\alpha_2^n\right) \\ c\left(\alpha_1^n-\alpha_2^n\right) & (a-\alpha_2)\alpha_2^n-(a-\alpha_1)\alpha_1^n \end{pmatrix} \\ A^n-\alpha_2^n I &= \frac{\alpha_1^n-\alpha_2^n}{\alpha_1-\alpha_2}(A-\alpha_2 I)\ \text{and}\ A^n-\alpha_1^n I=\frac{\alpha_1^n-\alpha_2^n}{\alpha_1-\alpha_2}(A-\alpha_1 I) \\ A^n &= \left(\frac{\alpha_1^n-\alpha_2^n}{\alpha_1-\alpha_2}\right)A-\alpha_1\alpha_2\left(\frac{\alpha_1^{n-1}-\alpha_2^{n-1}}{\alpha_1-\alpha_2}\right)I \end{aligned}$$

Proof. 1) $A^n=\alpha^{n-1}\begin{pmatrix} na-(n-1)\alpha & nb \\ nc & (n+1)\alpha-na \end{pmatrix}$
And
$A^n=\alpha^{n-1}\begin{pmatrix} na-(n-1)\alpha & nb \\ nc & nd+(n+1)\alpha-n(a+d) \end{pmatrix}$

4

$$A^n = \alpha^{n-1} \begin{pmatrix} na - (n-1)\alpha & nb \\ nc & nd + (n+1)\alpha - 2n\alpha \end{pmatrix}$$
$$A^n = \alpha^{n-1} \begin{pmatrix} na - (n-1)\alpha & nb \\ nc & nd + (1-n)\alpha \end{pmatrix}$$
$$A^n = \alpha^{n-1} \begin{pmatrix} na & nb \\ nc & nd \end{pmatrix} + \alpha^{n-1} \begin{pmatrix} -(n-1)\alpha & 0 \\ 0 & -(n-1)\alpha \end{pmatrix}$$
$$A^n = n\alpha^{n-1}A - (n-1)\alpha^n I$$

2)

$$A^n = \frac{1}{\alpha_1 - \alpha_2} \begin{pmatrix} a(\alpha_1^n - \alpha_2^n) - \alpha_1\alpha_2(\alpha_1^{n-1} - \alpha_2^{n-1}) & b(\alpha_1^n - \alpha_2^n) \\ c(\alpha_1^n - \alpha_2^n) & (\alpha_1^{n+1} - \alpha_2^{n+1}) - a(\alpha_1^n - \alpha_2^n) \end{pmatrix}$$

■

3 The exponential of any diagonalizable or non-diagonalizable matrix of order 2:

3.1 Introduction:

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ a 2X2 matrix and α_1 and α_2 solutions of $|A - \alpha I| = 0$.

In linear algebra, the Cayley–Hamilton theorem states that any endomorphism of a finite-dimensional vector space over any commutative field vanishes its own characteristic polynomial. In matrix terms, this means that if A is a square matrix of order and if $P(x) = |A - xI|$ is the characteristic polynomial of A, so $P(A) = 0$.

So, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$P(x) = |A - xI| = \begin{vmatrix} a - x & b \\ c & d - x \end{vmatrix} = (x-a)(x-d) - bc$

$P(x) = x^2 - (a+d)x + ad - bc = x^2 - Tr(A)x + |A|$

$P(A) = 0 \Longrightarrow$

$$A^2 - Tr(A)A + |A|I = 0$$

$\Longleftrightarrow$

$$A^2 = Tr(A)A - \det(A)I$$

$\Longrightarrow$

$$A^n = a_n A - b_n I$$

We know that

$$e^A = \sum_{n \geq 0} \frac{A^n}{n!}$$

And from $A^n = a_n A - b_n I$

$$e^A = rA + sI$$

Proposition 3 *If A and B commute so e^A and e^B commute and $e^A e^B = e^{A+B}$.*

$$AB = BA \Longrightarrow e^A e^B = e^B e^A = e^{A+B}$$

Proof. Obvious. ■

3.2 1st case : If $\alpha_1 \neq \alpha_2$.

We have seen: If $\alpha_1 \neq \alpha_2$ so $\forall n \in IN \setminus \{0,1\}$

$$A^n = \frac{1}{\alpha_1 - \alpha_2} \begin{pmatrix} (a-\alpha_2)\,\alpha_1^n - (a-\alpha_1)\,\alpha_2^n & b\,(\alpha_1^n - \alpha_2^n) \\ c\,(\alpha_1^n - \alpha_2^n) & (a-\alpha_2)\,\alpha_2^n - (a-\alpha_1)\,\alpha_1^n \end{pmatrix}$$

Remark 4 *If $\alpha_1 \neq \alpha_2$ so $\forall n \in IN$*

$$A^n = \frac{1}{\alpha_1 - \alpha_2} \begin{pmatrix} (a-\alpha_2)\,\alpha_1^n - (a-\alpha_1)\,\alpha_2^n & b\,(\alpha_1^n - \alpha_2^n) \\ c\,(\alpha_1^n - \alpha_2^n) & (a-\alpha_2)\,\alpha_2^n - (a-\alpha_1)\,\alpha_1^n \end{pmatrix}$$

Proof. $A^0 = I$ and

$$A^1 = \tfrac{1}{\alpha_1-\alpha_2} \begin{pmatrix} (a-\alpha_2)\,\alpha_1 - (a-\alpha_1)\,\alpha_2 & b\,(\alpha_1 - \alpha_2) \\ c\,(\alpha_1 - \alpha_2) & (a-\alpha_2)\,\alpha_2 - (a-\alpha_1)\,\alpha_1 \end{pmatrix}$$
$$A^1 = \begin{pmatrix} a & b \\ c & \frac{a\alpha_2 - \alpha_2^2 - a\alpha_1 + \alpha_1^2}{\alpha_1 - \alpha_2} \end{pmatrix}$$
$$A^1 = \begin{pmatrix} a & b \\ c & \alpha_1 + \alpha_2 - a \end{pmatrix}$$
$$A^1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A \ ■$$

Theorem 5 *Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ a 2x2 matrix and α_1 and α_2 solutions of $|A - \alpha I| = 0$, so*

$$\begin{aligned}
e^A &= \frac{1}{\alpha_1 - \alpha_2}\left[e^{\alpha_1}\begin{pmatrix} a-\alpha_2 & b \\ c & \alpha_1 - a \end{pmatrix} - e^{\alpha_2}\begin{pmatrix} a-\alpha_1 & b \\ c & \alpha_2 - a \end{pmatrix}\right] \\
e^A &= \frac{e^{\alpha_1} - e^{\alpha_2}}{\alpha_1 - \alpha_2}\,(A - \alpha_2 I) + e^{\alpha_2} I \\
e^A &= \frac{e^{\alpha_1} - e^{\alpha_2}}{\alpha_1 - \alpha_2}\,(A - \alpha_1 I) + e^{\alpha_1} I \\
e^A &= \frac{e^{\alpha_1} - e^{\alpha_2}}{\alpha_1 - \alpha_2} A + \frac{\alpha_1 e^{\alpha_2} - \alpha_2 e^{\alpha 1}}{\alpha_1 - \alpha_2} I
\end{aligned}$$

with $\alpha_1 \neq \alpha_2$ ($(a-d)^2 \neq -4bc$).

Proof. $e^A = \sum\limits_{n\geq 0} \frac{A^n}{n!}$

$$e^A = \sum_{n\geq 0} \frac{1}{n!}.\frac{1}{\alpha_1-\alpha_2} \begin{pmatrix} (a-\alpha_2)\,\alpha_1^n-(a-\alpha_1)\,\alpha_2^n & b\,(\alpha_1^n-\alpha_2^n) \\ c\,(\alpha_1^n-\alpha_2^n) & (a-\alpha_2)\,\alpha_2^n-(a-\alpha_1)\,\alpha_1^n \end{pmatrix}$$

since $\alpha_1 \neq \alpha_2$

$\Longrightarrow$

$$e^A = \frac{1}{\alpha_1-\alpha_2} \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}$$

with

$$\begin{cases} X = (a-\alpha_2) \sum\limits_{n\geq 0} \frac{1}{n!}\alpha_1^n - (a-\alpha_1) \sum\limits_{n\geq 0} \frac{1}{n!}\alpha_2^n \\ Y = b \sum\limits_{n\geq 0} \frac{1}{n!}\left(\alpha_1^n-\alpha_2^n\right) \\ Z = c \sum\limits_{n\geq 0} \frac{1}{n!}\left(\alpha_1^n-\alpha_2^n\right) \\ T = (a-\alpha_2) \sum\limits_{n\geq 0} \frac{1}{n!}\alpha_2^n - (a-\alpha_1) \sum\limits_{n\geq 0} \frac{1}{n!}\alpha_1^n \end{cases}$$

$$e^A = \frac{1}{\alpha_1-\alpha_2}. \begin{pmatrix} (a-\alpha_2)\,e^{\alpha_1}-(a-\alpha_1)\,e^{\alpha_2} & b\,(e^{\alpha_1}-e^{\alpha_2}) \\ c\,(e^{\alpha_1}-e^{\alpha_2}) & (a-\alpha_2)\,e^{\alpha_2}-(a-\alpha_1)\,e^{\alpha_1} \end{pmatrix}$$

Moreover

$a_{11} = (a-\alpha_2)\,e^{\alpha_1} - (a-\alpha_1)\,e^{\alpha_2}$
$a_{11} = (a-\alpha_2)\,e^{\alpha_1} - (a-\alpha_2)\,e^{\alpha_2} + (a-\alpha_2)\,e^{\alpha_2} - (a-\alpha_1)\,e^{\alpha_2}$
$a_{11} = (a-\alpha_2)\,(e^{\alpha_1}-e^{\alpha_2}) + (a-\alpha_2-a+\alpha_1)\,e^{\alpha_2}$
$a_{11} = (a-\alpha_2)\,(e^{\alpha_1}-e^{\alpha_2}) + (\alpha_1-\alpha_2)\,e^{\alpha_2}$

and

$a_{22} = (a-\alpha_2)\,e^{\alpha_2} - (a-\alpha_1)\,e^{\alpha_1}$
$a_{22} = (a-\alpha_2)\,e^{\alpha_2} - (a-\alpha_1)\,e^{\alpha_2} + (a-\alpha_1)\,e^{\alpha_2} - (a-\alpha_1)\,e^{\alpha_1}$
$a_{22} = [(a-\alpha_2)-(a-\alpha_1)]\,e^{\alpha_2} + (a-\alpha_1)\,(e^{\alpha_2}-e^{\alpha_1})$
$a_{22} = -\,(a-\alpha_1)\,(e^{\alpha_1}-e^{\alpha_2}) + (\alpha_1-\alpha_2)\,e^{\alpha_2}$

$\Longrightarrow$

$$e^A = \begin{pmatrix} \frac{(a-\alpha_2)(e^{\alpha_1}-e^{\alpha_2})+(\alpha_1-\alpha_2)e^{\alpha_2}}{\alpha_1-\alpha_2} & \frac{b(e^{\alpha_1}-e^{\alpha_2})}{\alpha_1-\alpha_2} \\ \frac{c(e^{\alpha_1}-e^{\alpha_2})}{\alpha_1-\alpha_2} & \frac{-(a-\alpha_1)(e^{\alpha_1}-e^{\alpha_2})+(\alpha_1-\alpha_2)e^{\alpha_2}}{\alpha_1-\alpha_2} \end{pmatrix}$$

$\Longrightarrow$

$e^A = \frac{e^{\alpha_1}-e^{\alpha_2}}{\alpha_1-\alpha_2} \begin{pmatrix} a-\alpha_2 & b \\ c & -a+\alpha_1 \end{pmatrix} + e^{\alpha_2} I$

$e^A = \frac{e^{\alpha_1}-e^{\alpha_2}}{\alpha_1-\alpha_2} \begin{pmatrix} a-\alpha_2 & b \\ c & d-(a+d)+\alpha_1 \end{pmatrix} + e^{\alpha_2} I$

$e^A = \frac{e^{\alpha_1}-e^{\alpha_2}}{\alpha_1-\alpha_2} \begin{pmatrix} a-\alpha_2 & b \\ c & d-(\alpha_1+\alpha_2)+\alpha_1 \end{pmatrix} + e^{\alpha_2} I$

since $\alpha_1+\alpha_2 = a+d \Longrightarrow$

$e^A = \frac{e^{\alpha_1}-e^{\alpha_2}}{\alpha_1-\alpha_2} \begin{pmatrix} a-\alpha_2 & b \\ c & d-\alpha_2 \end{pmatrix} + e^{\alpha_2} I$

$$e^A = \frac{e^{\alpha_1}-e^{\alpha_2}}{\alpha_1-\alpha_2}\left[A-\alpha_2 I\right] + e^{\alpha_2} I$$

idem for α_1, we have

$$e^A = \frac{e^{\alpha_1} - e^{\alpha_2}}{\alpha_1 - \alpha_2}\left[A - \alpha_1 I\right] + e^{\alpha_1} I$$

since $\frac{e^{\alpha_1}-e^{\alpha_2}}{\alpha_1-\alpha_2} = \frac{e^{\alpha_2}-e^{\alpha_1}}{\alpha_2-\alpha_1}$

$$e^A = \frac{e^{\alpha_1} - e^{\alpha_2}}{\alpha_1 - \alpha_2} A + \frac{\alpha_1 e^{\alpha_2} - \alpha_2 e^{\alpha_1}}{\alpha_1 - \alpha_2} I$$

hence the result. ■

Remark 6 *1)* $e^A = \frac{e^{\alpha_1}-e^{\alpha_2}}{\alpha_1-\alpha_2}A + \frac{\alpha_1 e^{\alpha_2}-\alpha_2 e^{\alpha 1}}{\alpha_1-\alpha_2}I.$

2) If $\alpha_1 = \alpha + i\beta$, *so* $\alpha_2 = \overline{\alpha}_1 = \alpha - i\beta \in \mathbb{C}\backslash IR$, $e^{\alpha_2} = e^{\alpha}e^{-i\beta} = \overline{(e^{\alpha_1})}$ *and* $\alpha_2 e^{\alpha 1} = \overline{(\alpha_1 e^{\alpha_2})}$

$\Longrightarrow \alpha_1 - \alpha_2 \in iIR$, $e^{\alpha_1} - e^{\alpha_2} \in iIR$ *and* $\alpha_1 e^{\alpha_2} - \alpha_2 e^{\alpha 1} \in iIR$

$\Longrightarrow \frac{e^{\alpha_1}-e^{\alpha_2}}{\alpha_1-\alpha_2} \in IR$ *and* $\frac{\alpha_1 e^{\alpha_2}-\alpha_2 e^{\alpha 1}}{\alpha_1-\alpha_2} \in IR.$

3.3 2nd case: If $\alpha_1 = \alpha_2$.

Remark 7

$$A^1 = A$$

Proof. $A^1 = \alpha^{1-1}\begin{pmatrix} a & b \\ c & (1+1)\alpha - a \end{pmatrix}$

$A^1 = \begin{pmatrix} a & b \\ c & 2\alpha - a \end{pmatrix}$

$A^1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$ ■

Theorem 8 *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *a 2X2 matrix such* $(a-d)^2 = -4bc$ *so the solutions of* $|A - \alpha I| = 0$ *are* $\alpha_1 = \alpha_2 = \alpha = \frac{a+d}{2}$ *and*

$$\begin{aligned} e^A &= e^{\frac{a+d}{2}}\left[A + \left(1 - \frac{a+d}{2}\right) I\right] \\ e^A &= e^{\alpha}\left[A + (1-\alpha) I\right] \end{aligned}$$

Proof. If $\alpha_1 = \alpha_2 = \alpha$, so $\forall n \geq 2$ $(n \in IN)$

$A^n = \alpha^{n-1}\begin{pmatrix} na - (n-1)\alpha & nb \\ nc & (n+1)\alpha - na \end{pmatrix}$

$A^n = \left(\frac{a+d}{2}\right)^{n-1}\begin{pmatrix} \frac{(n+1)a-(n-1)d}{2} & nb \\ nc & \frac{(n+1)d-(n-1)a}{2} \end{pmatrix}$,

with $\alpha = \frac{a+d}{2}$ and $(a+d)^2 = 4(ad - bc)$ $((a-d)^2 = -4bc)$

We have $e^A = \sum\limits_{n\geq 0} \frac{1}{n!}A^n = \sum\limits_{n\geq 1}\frac{1}{n!}A^n + I$

so, $e^A = \sum_{n\geq 1} \frac{1}{n!}\alpha^{n-1} \begin{pmatrix} na-(n-1)\alpha & nb \\ nc & (n+1)\alpha - na \end{pmatrix} + I$

$e^A = \sum_{n\geq 1} \frac{1}{n!} \begin{pmatrix} an\alpha^{n-1}-(n-1)\alpha^n & bn\alpha^{n-1} \\ cn\alpha^{n-1} & (n+1)\alpha^n - an\alpha^{n-1} \end{pmatrix} + I$

$= \begin{pmatrix} ae^{\alpha} - \sum_{n\geq 1}\left(\frac{1}{(n-1)!}\alpha^n - \frac{1}{n!}\alpha^n\right) & be^{\alpha} \\ ce^{\alpha} & \sum_{n\geq 1}\left(\frac{1}{(n-1)!}\alpha^n + \frac{1}{n!}\alpha^n\right) - ae^{\alpha} \end{pmatrix} + I$

$e^A = \begin{pmatrix} ae^{\alpha} - (\alpha e^{\alpha} - e^{\alpha} + 1) & be^{\alpha} \\ ce^{\alpha} & (\alpha e^{\alpha} + e^{\alpha} - 1) - ae^{\alpha} \end{pmatrix} + I$

$e^A = \begin{pmatrix} ae^{\alpha} - \alpha e^{\alpha} + e^{\alpha} - 1 & be^{\alpha} \\ ce^{\alpha} & \alpha e^{\alpha} + e^{\alpha} - 1 - ae^{\alpha} \end{pmatrix} + I$

$e^A = \begin{pmatrix} ae^{\alpha} - \alpha e^{\alpha} + e^{\alpha} & be^{\alpha} \\ ce^{\alpha} & \alpha e^{\alpha} + e^{\alpha} - ae^{\alpha} \end{pmatrix}$

$e^A = e^{\alpha} \begin{pmatrix} a-\alpha+1 & b \\ c & \alpha+1-a \end{pmatrix}$

$e^A = e^{\alpha} \left[A + \begin{pmatrix} -\alpha+1 & 0 \\ 0 & \alpha+1-a-d \end{pmatrix} \right]$

$e^A = e^{\alpha} \left[A + \begin{pmatrix} -\alpha+1 & 0 \\ 0 & \alpha+1-2\alpha \end{pmatrix} \right]$

since $\alpha = \frac{a+d}{2}$

$e^A = e^{\alpha} \left[A + \begin{pmatrix} -\alpha+1 & 0 \\ 0 & 1-\alpha \end{pmatrix} \right]$

$$e^A = e^{\alpha}\left[A + (1-\alpha) I\right]$$

$$e^A = e^{\frac{a+d}{2}} \left[A + \left(1 - \frac{a+d}{2}\right) I\right]$$

■

Remark 9

$$e^{(A-\alpha I)} - (A - \alpha I) = I$$

Theorem 10 *(Generalization):*

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *a 2X2 matrix and* α_1 *and* α_2 *solutions of* $|A-\alpha I| = 0$, *so*

$$e^A = e^{\alpha_2}\left[\sum_{n\geq 1} \frac{(\alpha_1-\alpha_2)^{n-1}}{n!}(A - \alpha_2 I) + I\right]$$

Proof. 1st case: if $\alpha_1 = \alpha_2 = \alpha$, so

$e^{\alpha_2}\left[\sum_{n\geq 1} \frac{(\alpha_1-\alpha_2)^{n-1}}{n!}(A-\alpha_2 I) + I\right] = e^{\alpha}\left[(A-\alpha I) + I\right]$

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such $\sum\limits_{n\geq 1} \frac{(\alpha_1-\alpha_2)^{n-1}}{n!} = 1 + (\alpha_1 - \alpha_2) \sum\limits_{n\geq 2} \frac{(\alpha_1-\alpha_2)^{n-2}}{n!} = 1$

$\Longrightarrow e^{\alpha_2} \left[\sum\limits_{n\geq 1} \frac{(\alpha_1-\alpha_2)^{n-1}}{n!} (A - \alpha_2 I) + I \right] = e^{\alpha} [A + (1-\alpha) I] = e^A$

2d case: if $\alpha_1 \neq \alpha_2$, so

$e^{\alpha_2} \left[\sum\limits_{n\geq 1} \frac{(\alpha_1-\alpha_2)^{n-1}}{n!} (A - \alpha_2 I) + I \right] = e^{\alpha_2} \left[\left(\frac{e^{\alpha_1-\alpha_2}-1}{\alpha_1-\alpha_2} \right) (A - \alpha_2 I) + I \right]$

since $\sum\limits_{n\geq 1} \frac{(\alpha_1-\alpha_2)^{n}}{n!} = e^{\alpha_1-\alpha_2} - 1$

$\Longrightarrow e^{\alpha_2} \left[\sum\limits_{n\geq 1} \frac{(\alpha_1-\alpha_2)^{n-1}}{n!} (A - \alpha_2 I) + I \right] = e^{\alpha_2} \left(\frac{e^{\alpha_1-\alpha_2}-1}{\alpha_1-\alpha_2} \right) (A - \alpha_2 I) + e^{\alpha_2} I$

$\Longrightarrow e^{\alpha_2} \left[\sum\limits_{n\geq 1} \frac{(\alpha_1-\alpha_2)^{n-1}}{n!} (A - \alpha_2 I) + I \right] = \left(\frac{e^{\alpha_1}-e^{\alpha_2}}{\alpha_1-\alpha_2} \right) (A - \alpha_2 I) + e^{\alpha_2} I$

$\Longrightarrow e^{\alpha_2} \left[\sum\limits_{n\geq 1} \frac{(\alpha_1-\alpha_2)^{n-1}}{n!} (A - \alpha_2 I) + I \right] = \left(\frac{e^{\alpha_1}-e^{\alpha_2}}{\alpha_1-\alpha_2} \right) A + \left(e^{\alpha_2} - \alpha_2 \frac{e^{\alpha_1}-e^{\alpha_2}}{\alpha_1-\alpha_2} \right) I$

$\Longrightarrow e^{\alpha_2} \left[\sum\limits_{n\geq 1} \frac{(\alpha_1-\alpha_2)^{n-1}}{n!} (A - \alpha_2 I) + I \right] = \left(\frac{e^{\alpha_1}-e^{\alpha_2}}{\alpha_1-\alpha_2} \right) A + \left(\frac{(\alpha_1-\alpha_2)e^{\alpha_2}-\alpha_2(e^{\alpha_1}-e^{\alpha_2})}{\alpha_1-\alpha_2} \right) I$

$\Longrightarrow e^{\alpha_2} \left[\sum\limits_{n\geq 1} \frac{(\alpha_1-\alpha_2)^{n-1}}{n!} (A - \alpha_2 I) + I \right] = \left(\frac{e^{\alpha_1}-e^{\alpha_2}}{\alpha_1-\alpha_2} \right) A + \left(\frac{\alpha_1 e^{\alpha_2}-\alpha_2 e^{\alpha_1}}{\alpha_1-\alpha_2} \right) I =$

e^A ■

Proposition 11

$$e^A = I \Longrightarrow (A = O \quad or\ A^2 = -4k^2\pi^2 I\ )$$

Proof. Let α be an eigenvalue of A, so

$\exists X \in IR^2$ such $X \neq (0,0)$ and $AX = \alpha X$

$e^A X = X \Longrightarrow \sum\limits_{n\geq 0} \frac{A^n}{n!} X = X$

$\Longrightarrow \sum\limits_{n\geq 0} \frac{\alpha^n}{n!} X = X$

$\Rightarrow e^{\alpha} X = X$

$\Longrightarrow e^{\alpha} = 1$

such $X \neq (0,0)$

1st case: If $\alpha \in IR$ so $e^{\alpha} = 1 \Longrightarrow \alpha = 0$

and the characteristic polynomial of A will have to be $P(x) = x^2 - Tr(A)\, x$

So $\alpha_1 = 0$ and $\alpha_2 = Tr(A) \in IR$.

Applying the same calculation as before we will obtain

$\alpha_2 = Tr(A) = 0$

$\Longrightarrow P(x) = x^2$

$\Longrightarrow$

$$A^2 = O$$

since $P(A) = O$ (Applying the Cayley–Hamilton theorem)

$\Longrightarrow$

$$A^n = O, \quad \forall n \in IN^* - \{1\}$$

$e^A = I \Longrightarrow \sum_{n\geq 0} \frac{A^n}{n!} = I$
$\Longrightarrow I + A = I$
$\Longrightarrow$

$$A = O$$

2d case: If $\alpha \in \mathbb{C}\backslash IR$ so $\overline{\alpha} \in \mathbb{C}\backslash IR$ is also an eigenvalue of A.
So $e^{\alpha} = 1 \Longrightarrow \alpha = 2ik\pi$
$\alpha \in \mathbb{C}\backslash IR \Longrightarrow k \neq 0$
$k \in \mathbb{Z}^*$ $(k \neq 0)\Longrightarrow$ A is diagonalisable and the characteristic polynomial of it, is
$P(x) = x^2 - (\alpha + \overline{\alpha})\, x + \alpha\overline{\alpha}$
$\Longrightarrow P(x) = x^2 + 4k^2\pi^2$
$\Longrightarrow A^2 = -4k^2\pi^2 I$ ■

Conclusion 12 *We conclude that the final formula we obtained for "The exponential of any (diagonalizable or non-diagonalizable) matrix of order 2"*

$$e^A = e^{\alpha_2}\left[\sum_{n\geq 1} \frac{(\alpha_1 - \alpha_2)^{n-1}}{n!}(A - \alpha_2 I) + I\right]$$

is as general as those given for the n-th Power of 2x2 Matrix: (see [1], [2])

$$A^n = \begin{pmatrix} a\sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - \sum_{k=0}^{n-2} \alpha_1^{k+1}\alpha_2^{n-1-k} & b\sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} \\ c\sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} & \sum_{k=0}^{n} \alpha_1^k \alpha_2^{n-k} - a\sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} \end{pmatrix}$$

and without discussing whether the eigenvalues are equal or not, thus, we were able to combine all cases into a single "New" formula.

Competing Interests
As the sole author of this work, I declare that I have no conflict of interest with anyone.

Authors' Contributions
I am the sole author and contributor to the writing of this article. I have read and approved the final manuscript.

The type of article
Original Research Article.

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