A Finite Capacity Single Server Queueing System with Additional Optional Service

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Abstract

This paper contracts with a finite-capacity Markovian queueing structure with state-dependent arrival and service rates [4]. It is expected that the customers may seek second optional additional service after completion of the first essential service. This model is defined using the infinitesimal generator matrix and for the analysis, the group generalised inverse of the infinitesimal generator is used. Using the group generalised inverses[1], the probabilities associated with the steady state are obtained analytically. Some performance measures are derived. Also, some numerical illustrations are provided.

Keywords: A Markovian queueing structure with limited capacity, State-dependent rates[4], Infinitesimal generator matrix, Group generalised inverse, Steady state probabilities, and Performance measures.

1. Introduction

In the literature, many researchers work on queueing models with infinite waiting space. However, in practice, in waiting line situations, the capacity of the line is finite. Since the waiting space is finite, the arriving customers leave the system without enter the waiting line, if the line is full, such a model is called the loss model. This type of model was first investigated by Erlang in 1917. Some earlier researchers on this area are Fortet (1948), Vaulot (1954), Takacs(1969), Jagerman (1974), Harel (1987), Berezner et al.,(1995) and Kalyanaraman and Pattabiraman (2010).

Also, in the literature, the arrival rate and service rate are constant in many works on queueing systems. However, in many real-life situations, the arrival and service rates need not be constant; the rates are state-dependent, that is, λ_n and μ_n respectively, where n is the number of customers in the system(Cox and Smith(1961)). Based on physical situations, some special cases that received considerable attention in the literature are

$$\lambda_n = \begin{cases} n\lambda, & n > 0 \\ \dfrac{\lambda}{n+1}, & n \geq 0 \end{cases}$$

$$\mu_n = \begin{cases} n\mu, & n \geq 1 \\ \mu, & n \geq 1 \end{cases}$$

Notable works in this line are Conway and Maxwell (1961), Harris (1967), Scott (1970), Natvig (1973),Hadidi (1974), Conolly (1975), Boxma et.al (2005), Kalyanaraman and Pattabiraman (2011) and Kalyanaraman and Sundaramoorthy (2019).

Queueing model through main services have been considered in many studies. However, in many real situations, some customers may need the main service and some subsidiary services as well. Madan in 2000, presented the idea of a second optional service. After the pioneering work of Madan many authors including Medhi (2002), Wang (2004), Choudhury et.al. (2009) and others working on queueing models with a second optional service.

In this article, we analysed a finite-capacity Markovian queueing system with state-dependent arrival and service rates[4]. After the end of service (essential service) the customer leaves the system with probability 1-r or the customer may demand additional service (optional service) with probability r (0<r<1).

An application of our model is in the subsequent automobile repair garage for cars. Consider a mechanic of an automobile repair garage, the arrival of cars forms a random process. The mechanic is responsible for routine maintenance (essential service) of a car at one time (such as the routine maintenance every 1000 km, 2000km, etc.). Few vehicles may need a tire, windshield wiper, or battery spare (second optional service).

This model is defined using an infinitesimal generator matrix. The above-defined model has been analysed using the technique of group generalized inverses. Hunter (1969) recognised that a square medium G processes the group inverse, whenever G and G^2 have the same rank. Some notable works in this area are Adi-Ben. Israel and Greville (1974), Boullion and Odell (1971) and Campbell and Meyer (1979). Meyer (1975) gave a formula for a group inverse of an infinitesimal generator of m-state ergodic processes. Using the group inverse he obtained the fixed probability vector. Kemney and Snell (1960) and Hunter (1969) have obtained the mean first passage time matrix for m-state ergodic processes.

This paper is formatted as follows: In sections 2 and 3, the method and the corresponding analysis are given. In section 4, the model definition is given. In section 5, the mathematical definition of the queueing system and the analysis are given. In section 6, some performance measures are given. In section 7, some numerical illustrations are provided. Finally, in section 8, a conclusion is given.

2. The Method

Let $(X_n, Y_n),\ n \geq 0$ be a Markov process on the state $S = \{(n,j): 0 \leq n \leq N,\ 1 \leq j \leq a_n\}$with the following bock tridiagonal infinitesimal generator.

$$Q = \begin{pmatrix} B_0 & A_0 & 0 & 0 & \dots & \dots & 0 & 0 \\ C_1 & B_1 & A_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & C_2 & B_2 & A_2 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots & C_{N-1} & B_{N-1} & A_{N-1} \\ 0 & 0 & 0 & \dots & \dots & 0 & C_N & B_N \end{pmatrix}$$

Where $B_0, B_1, \dots, B_N$ are square matrices of order $a_0, a_1, \dots, a_N$ respectively. Their diagonal elements are strictly negative, the other elements are non-negative. The matrices $A_0, A_1, \dots, A_{N-1}$,$C_1, C_2, \dots, C_N$ are rectangular matrices and non-negative. The row sums of Q are equal to 0. That is,

$$B_0 e + A_0 e = 0$$
$$C_i e + B_i e + A_i e = 0: 1 \leq i \leq N-1,$$
$$C_N\, e + B_N e = 0$$

Where e denotes the column vector and unit elements.

For the determination of the stationary probability distribution, the following realization of the Markov chain is useful. Observe the process Q during the interval of time spent at the level$n$, before the original process enters the level $n+1$ for the first time. DenoteP_n, be the realization of the process. The state space of P_n is $S_n = \{(n,j): 1 \leq j \leq a_n\}$. All $P_n,\ 0 \leq n \leq N-1$ are transient Markov Chains. The process p_N is the realization of the process Q with state space $S_N = \{(N,j): 1 \leq j \leq a_N\}$, it is an ergodic Markov chain. Denote Q_n as the infinitesimal generator of the process$P_n,\ 0 \leq n \leq N$. Let $W = (w_0, w_1, w_2, \dots, w_N)$ be the probability vector, where $W_n$be the probability that there are n customers in the system.

3. The Algorithm

Based on the method described in the above section the following algorithm is incorporated to find the analytic solution for the model defined in section 2.

Based on the method described in the above sub-section the following algorithm is proposed to solve the model defined in this paper:

Step 1 Write $R = \begin{pmatrix} U & c \\ d' & \alpha \end{pmatrix}$ $R = -Q$, U is $(m-1)\times(m-1)$ matrix

Step 2: Check the rank of$R$ = rank of R^2, if it is true $R^{\#}$ exists.

Step 3: Calculate $h' = d'U^{-1}$ and $\beta = 1 - h'j$ where $\beta$is non-zero.

Step 4: Calculate $W' = \frac{1}{\beta}[-h', 1]$.

The Model

The queueing model discussed in this paper is a single server queue with a finite waiting line of size N. The arrival process follows a Poisson process with a rate λ_n and service time of successive customers follow a negative exponential distribution with a rate μ_n, where n the number of customers in the system at the period of arrival and service respectively. After the end of service (essential service) the customer exits the system with probability 1-r or the customer may demand additional service (optional service) with probability r (0<r<1). The optional service follows a negative exponential with rate μ. Also on arrival, if the system is full of N customers, then the arrival leaves the system.

The Markov chain related to the model defined above is $\{(X_n, Y_n): n \geq 0\}$ with state space.$S = \{(i,j): 0 \leq i \leq N,\ j = a_0, a_1, a_2\}$.

The Q matrix is.

$$Q = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \mu_2 + r\mu & -(\lambda_2 + \mu_2 + r\mu) & \lambda_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \mu_{N-1} + r\mu & -(\lambda_{N-1} + \mu_{N-1} + r\mu) & \lambda_{N-1} \\ 0 & 0 & 0 & \cdots & \cdots & 0 & \mu_N + r\mu & -(\mu_N + r\mu) \end{bmatrix}$$

Where $a_0 = \{0\},\ a_1 = \{1,2,3, \dots,\ N-1\},\ a_2 = \{N\}$.

Define $R = \begin{pmatrix} U & c \\ d' & \alpha \end{pmatrix}$, where $R = -Q$, U is an $N \times N$ matrix, corresponding to the states $\{0,1,2, \dots, N\}$.

4. The Mathematical Model and Study

The model defined in the article can be identified utilizing the above construction. For the solution, we use the technique of group inverse (Kalyanaraman & Pattabiraman (2010)). The Markov chain related to this article is $\{(X_n, Y_n): n \geq 0\}$ with state space.$S = \{(i,j): 0 \leq i \leq N,\ j = a_0, a_1, a_2\}$.

To put on the process of unit 4, we take $R = \begin{pmatrix} U & c \\ d' & \alpha \end{pmatrix}$ where.

$R = -Q,$

The Q matrix is.

$$Q = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \mu_2 + r\mu & -(\lambda_2+\mu_2+r\mu) & \lambda_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \mu_{N-1}+r\mu & -(\lambda_{N-1}+\mu_{N-1}+r\mu) & \lambda_{N-1} \\ 0 & 0 & 0 & \cdots & \cdots & 0 & \mu_N + r\mu & -(\mu_N + r\mu) \end{bmatrix}$$

Where $a_0 = \{0\},\ a_1 = \{1,2,3,\dots,\ N-1\},\ a_2 = \{N\}$

$$U = \begin{pmatrix} \lambda_0 & -\lambda_0 & 0 & 0 & 0 & \dots & 0 & 0 \\ -(\mu_1 + r\mu) & (\lambda_1 + \mu_1) & -\lambda_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -(\mu_2 + r\mu) & (\lambda_2 + \mu_2 + r\mu) & -\lambda_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & -(\mu_3 + r\mu) & (\lambda_3 + \mu_3 + r\mu) & -\lambda_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \dots & \dots & -(\mu_{N-1} + r\mu) & (\lambda_{N-1} + \mu_N + r\mu) \end{pmatrix}$$

$$c = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \dots \\ \dots \\ -\lambda_{N-1} \end{pmatrix}$$

$$d' = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -(\mu_N + r\mu) \end{bmatrix} \quad \alpha = (\mu_N + r\mu)$$

We examine the above model by means of technique described in Unit 4.

$$U^{-1} = (u_{jj})_{(N-1)\times(N-1)}$$

Where

$$u_{ij} = \frac{1}{\lambda_{j-1}}\left[1+\frac{\mu_1}{\lambda_1}\left(1+\sum_{k=j}^{n-3}\prod_{l=j}^{k}\frac{\mu_{l+1}+r\mu}{\lambda_{l+1}}\right)\right] \quad \text{for i,, j}$$

$$u_{ij} = \frac{1}{\lambda_{j-1}}\left[\frac{\mu_1}{\lambda_1}\left(1+\sum_{k=j}^{n-3}\prod_{l=j}^{k}\frac{\mu_{l+1}+r\mu}{\lambda_{l+1}}\right)\right] \text{for } i > j$$

The unique fixed probability vector $W' = \frac{1}{\beta}(W_0, W_1, ..., W_N)$

$$K = (\mu_1+\lambda_0)\prod_{j=2}^{N}(\mu_j+r\mu) + \sum_{k=1}^{N-1}\prod_{j=0}^{k-1}\lambda_j\prod_{j=k+2}^{N}(\mu_j+r\mu) + \prod_{j=0}^{N-1}\lambda_j$$

$$\beta = \frac{1}{\prod_{j=0}^{N-1}\lambda_j}\left[(\mu_1+\lambda_0)\prod_{j=2}^{N}(\mu_j+r\mu) + \sum_{k=1}^{N-1}\prod_{j=0}^{k-1}\lambda_j\prod_{j=k+2}^{N}(\mu_j+r\mu) + \prod_{j=0}^{N-1}\lambda_j\right]$$

Where,

$$W_0 = \frac{1}{K}\left[(\mu_j+\lambda_0)\prod_{j=2}^{N}(\mu_j+r\mu)\right]$$

$$W_i = \frac{1}{K}\left[\prod_{j=0}^{k-1}\lambda_j\prod_{j=k+1}^{N}(\mu_j+r\mu)\right]$$

$$W_N = \frac{1}{K}\prod_{j=0}^{N-1}\lambda_j$$

5. Some Performance measures.

The Mean number of customers in the system is $L = \sum_{n=1}^{N} nW_n$

The probability that the server is idle is W_0

The Blocking probability $p_\beta = 1 - W_n$

6. Numerical Analysis

In this segment, we examine three special models namely, Model I, Model II, and Model III related to model in unit 3. In this study, we take $\lambda_n = \frac{1}{n+1}, \mu_n = \frac{1}{2n}$ and vary for $N = 5$(Model I) $N = 7$(Model II) and $N = 15$ (Model III).

Model I: $M^{(n)}/M^{(n)}/1/5$

$$\lambda_0 = 1: \lambda_1 = 0.5: \lambda_2 = 0.333: \lambda_3 = 0.25: \lambda_4 = 0.2,$$

Optional service with $r = 0.5$ and second service follows exponential with $\mu = 0.5$

$\mu_1 = 0.5,\ \mu_2 = 0.25,\ \mu_3 = 0.167,\ \mu_4 = 0.125,\ \mu_5 = 0.10$, then,

$$R = -Q = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -0.75 & 1.25 & -0.5 & 0 & 0 & 0 \\ 0 & -0.5 & 0.833 & -0.333 & 0 & 0 \\ 0 & 0 & -0.417 & 0.667 & -0.25 & 0 \\ 0 & 0 & 0 & -0.375 & 0.575 & -0.2 \\ 0 & 0 & 0 & 0 & -0.35 & 0.35 \end{pmatrix}$$

$$\mathrm{U}^{-1} = \begin{pmatrix} 15.553 & 19.404 & 17.404 & 11.5 & 5 \\ 14.553 & 19.404 & 17.404 & 11.5 & 5 \\ 13.053 & 17.404 & 17.404 & 11.5 & 5 \\ 10.801 & 14.401 & 14.401 & 11.5 & 5 \\ 7.044 & 9.392 & 9.392 & 7.5 & 5 \end{pmatrix}$$

Rank of (Q) =rank $(Q^2) = 5$

$d' = (0 \quad 0 \quad 0 \quad 0 \quad -0.35)$

$h' = (-2.465 \quad -3.265 \quad -3.265 \quad -2.625 \quad -1.75)$

$\beta = 11.745$

$W' = (0.21 \quad 0.278 \quad 0.278 \quad 0.223 \quad 0.149 \quad 0.085)$

$L = 2.067$

$W_0 = 0.21$

The blocking probability p_β =0.915

Model II: $M^{(n)}/M^{(n)}/1/7$

$\lambda_0 = 1: \lambda_1 = 0.5: \lambda_2 = 0.333: \lambda_3 = 0.25: \lambda_4 = 0.2:$
$\lambda_5 = 0.167: \lambda_6 = 0.143$

Optional service with $r = 0.2$ and second service follows exponential with $\mu = 0.2$

$\mu_1 = 0.5: \mu_2 = 0.25: \mu_3 = 0.167: \mu_4 = 0.125:$
$\mu_5 = 0.10: \mu_6 = 0.083: \mu_7 = 0.071$

then,

$$R = -Q = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.54 & 1.04 & -0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.29 & 0.623 & -0.333 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.207 & 0.457 & -0.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.165 & 0.365 & -0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.14 & 0.307 & -0.167 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.123 & 0.266 & -0.143 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.111 & 0.111 \end{pmatrix}$$

$$U^{-1} = \begin{pmatrix} 5.444 & 8.229 & 10.74 & 12.446 & 12.797 & 11.139 & 6.993 \\ 4.444 & 8.229 & 10.74 & 12.446 & 12.797 & 11.139 & 6.993 \\ 3.364 & 6.229 & 10.74 & 12.446 & 12.797 & 11.139 & 6.993 \\ 2.423 & 4.487 & 7.737 & 12.446 & 12.797 & 11.139 & 6.993 \\ 1.644 & 3.045 & 5.25 & 8.446 & 12.797 & 11.139 & 6.993 \\ 1.002 & 1.855 & 3.199 & 5.146 & 7.797 & 11.139 & 6.993 \\ 0.463 & 0.858 & 1.479 & 2.38 & 3.605 & 5.151 & 6.993 \end{pmatrix}$$

Rank of (Q) =rank $(Q^2) = 7$

$d' = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -0.111)$

$h' = (-0.051 \quad -0.095 \quad -0.164 \quad -0.264 \quad -0.4 \quad -0.572 \quad -0.776)$

$\beta = 3.322$

$W' = (0.015 \quad 0.029 \quad 0.049 \quad 0.079 \quad 0.12 \quad 0.172 \quad 0.234 \quad 0.301)$

$L = 5.944 \, , W_0 = 0.015$

The blocking probability $P_\beta = 0.699$

Model III: $M^{(n)}/M^{(n)}/1/10$

$\lambda_0 = 1 : \lambda_1 = 0.5 : \lambda_2 = 0.333 : \lambda_3 = 0.25 : \lambda_4 = 0.2 :$
$\lambda_5 = 0.167 : \lambda_6 = 0.143 : \lambda_7 = 0.125 : \lambda_8 = 0.111 : \lambda_9 = 0.1 : \lambda_{10} = 0.091 :$

Optional service with $r = 0.2$ and second service follows exponential with $\mu = 0.2$

$\mu_1 = 0.5 : \mu_2 = 0.25 : \mu_3 = 0.167 : \mu_4 = 0.125 :$
$\mu_5 = 0.10 : \mu_6 = 0.083 : \mu_7 = 0.071, \mu_8 = 0.063 :$
$\mu_9 = 0.056 : \mu_{10} = 0.05 : \mu_{11}$

then,

$$R = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.54 & 1.040 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.290 & 0.623 & -0.333 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.207 & 0.457 & -0.250 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.165 & 0.365 & -0.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.140 & 0.307 & -0.167 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.123 & 0.266 & -0.143 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.111 & 0.236 & -0.125 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.103 & 0.214 & -0.111 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.096 & 0.196 & -0.1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.090 & 0.090 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.54 & 1.040 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.290 & 0.623 & -0.333 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.207 & 0.457 & -0.250 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.165 & 0.365 & -0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.140 & 0.307 & -0.167 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.123 & 0.266 & -0.143 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.111 & 0.236 & -0.125 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.103 & 0.214 & -0.111 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.096 & 0.196 \end{pmatrix}$$

$$U^{-1} = \begin{pmatrix} 6.6033 & 10.3765 & 14.4422 & 18.4021 & 21.8214 & 24.0306 & 24.4968 & 22.5499 & 17.6577 & 10.0000 \\ 5.6033 & 10.3765 & 14.4422 & 18.4021 & 21.8214 & 24.0306 & 24.4968 & 22.5499 & 17.6577 & 10.0000 \\ 4.5233 & 8.3765 & 14.4422 & 18.4021 & 21.8214 & 24.0306 & 24.4968 & 22.5499 & 17.6577 & 10.0000 \\ 3.5827 & 6.6347 & 11.4392 & 18.4021 & 21.8214 & 24.0306 & 24.4968 & 22.5499 & 17.6577 & 10.0000 \\ 2.8040 & 5.1925 & 8.9527 & 14.4021 & 21.8214 & 24.0306 & 24.4968 & 22.5499 & 17.6577 & 10.0000 \\ 2.1615 & 4.0028 & 6.9013 & 11.1021 & 16.8214 & 24.0306 & 24.4968 & 22.5499 & 17.6577 & 10.0000 \\ 1.6229 & 3.0053 & 5.1816 & 8.3357 & 12.6298 & 18.0425 & 24.4968 & 22.5499 & 17.6577 & 10.0000 \\ 1.1596 & 2.1474 & 3.7024 & 5.9561 & 9.0244 & 12.8920 & 17.5038 & 22.5499 & 17.6577 & 10.0000 \\ 0.7482 & 1.3856 & 2.3889 & 3.8431 & 5.8228 & 8.3183 & 11.2940 & 14.5499 & 17.6577 & 10.0000 \\ 0.3665 & 0.6787 & 1.1701 & 1.8823 & 2.8520 & 4.0743 & 5.5317 & 7.1265 & 8.6486 & 10.0000 \end{pmatrix}$$

Rank of (Q) =rank (Q^2) = 10

d'=$[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -0.096]$

h'
$= [-0.0352 \quad -0.0652 \quad -0.1123 \quad -0.1807 \quad -0.2738 \quad -0.3911 \quad -0.5310 \quad -0.6842 \quad -0.8303 \quad -0.9600]$

$$\beta = 5.0638$$

$$W = [0.007 \quad 0.0129 \quad 0.0222 \quad 0.0357 \quad 0.0541 \quad 0.0772 \quad 0.1049 \quad 0.1351 \quad 0.1640 \quad 0.1896 \quad 0.1975]$$

$L = 7.3374$

$W_0 = 0.007$

The Blocking probability $P_B = 0.8025$

7. Conclusion

This paper deals with a finite-capacity Markovian queueing system with state-dependent arrival and service rates[4]. After completion of the first essential service, the customer may demand additional optional services. We define the model using an infinitesimal generator matrix and for the analysis, we are applying the group generalised inverse of the infinitesimal generator matrix. Using the group generalised inverse the steady-state probabilities are obtained analytically. Some performance measures are derived. Also, we provide some numerical illustrations to show the practical applicability of the model.

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