Degenerate soliton solutions of first kind KdV equation with variable coefficients using Jacobi Elliptic Functions

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Abstract:

This paper explores various solitary and periodic wave solutions for nonlinear evolution equations with ordinary and variable coefficients. Utilizing the Jacobi Elliptic Functions, the study proposes a method incorporating an auxiliary equation and function transformation to construct new soliton-like and triangular wave solutions for first kind KdV equation. The methodology involves transforming the nonlinear evolution equation into a solvable form, leading to solutions expressed through Jacobi elliptic functions. These findings extend existing research on material property descriptions, providing exact solutions to nonlinear equations with variable coefficients, which are crucial for accurately modeling substance movement transformation systems.

Key words: Jacobi elliptic functions, nonlinear evolution equations, Degenrate soliton solutions, Forcible terms

1. Introduction :

Numerous solitary and periodic wave solutions for nonlinear evolution equations with constant coefficients have been proposed. The use of Jacobi Elliptic Functions is a highly effective technique for finding exact solutions to partial differential equations with forcing terms [1-10]. However, nonlinear evolution equations with constant coefficients only provide an approximate representation of the material transformation system. To better describe material properties, researchers have studied nonlinear evolution equations with variable coefficients, yielding significant research advancements [11-17] . This paper introduces a method involving an auxiliary equation with a function transformation. Using this approach, we construct new degenerate soliton-like solutions and triangular function wave solutions for first kind KdV equation with variable coefficients.

2. Method and Application Procedure :

Let a nonlinear evolution equation with variable coefficients ia as follows:

$$H(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0. \tag{1}$$

using transformation $u(x,t) = u(\xi)$, and $\xi = p(t)x + q(t)$, where $p(t)$ and $q(t)$ are functions of t and to be determined, and assume the solutions of Eq. (1) to be in the following form:

$$u(x,t) = g_0(t) + g_1(t)z(\xi), \tag{2}$$

Where $g_0(t)$ and $g_1(t)$ are functions of t and to be determined, and $z(\xi)$ is determined by the following auxiliary equation

$$\left(\frac{dz(\xi)}{d\xi}\right)^2 = az(\xi) + bz^2(\xi) + cz^3(\xi), \quad (3)$$

and we will obtain the solutions of Eq. (3) as follows:

when $a = 4, b = -4(1 + k^2)$, and $c = 4k^2$, then $z(\xi) = sn^2(\xi, k)$; (4)

when $a = -4(-1 + k^2), b = 4(-1 + 2k^2)$, and $c = -4k^2$, then $z(\xi) = cn^2(\xi, k)$; (5)

when $a = 4(-1 + k^2), b = -4(-2 + k^2)$, and $c = -4$then $z(\xi) = dn^2(\xi, k)$; (6)

when $a = 4k^2, b = -4(1 + k^2)$, and $c = 4$, then $z(\xi) = ns^2(\xi, k)$; (7)

when $a = -4k^2, b = 4(-1 + 2k^2)$, and $c = -4(-1 + k^2)$, then $z(\xi) = nc^2(\xi, k)$; (8)

when $a = -4, b = -(-2 + k^2)$, and $c = 4(-1 + k^2)$, then $z(\xi) = nd^2(\xi, k)$; (9)

when $a = 4, b = -4(-2 + k^2)$, and $c = -4(-1 + k^2)$ then $z(\xi) = sc^2(\xi, k)$; (10)

when $a = 4, b = 4(-1 + 2k^2)$, and $c = 4k^2(-1 + k^2)$, then $z(\xi) = sd^2(\xi, k)$; (11)

when $a = -4(-1 + k^2), b = -4(-2 + k^2)$, and c = 4, then $z(\xi) = cs^2(\xi, k)$; (12)

when $a = 4, b = -4(-1 + k^2)$, and $c = 4k^2$, then $z(\xi) = cd^2(\xi, k)$; (13)

when $a = 4k^2(-1 + k^2)^2, b = 4(-1 + k^2)$, and c = 4 then $z(\xi) = ds^2(\xi, k)$; (14)

when $a = 4k^2, b = -4(1 + k^2)$, and c = 4 then $z(\xi) = dc^2(\xi, k)$; (15)

when $a = -(1 - k^2)^2, b = 2(1 + k^2)$, and $c = -1$, then $z(\xi) = (kcn(\xi, k) \pm dn(\xi, k)^2$; (16)

when $a = 1, b = -2(-1 + 2k^2)$, and c = 1, then $z(\xi) = (ns(\xi, k) \pm cs(\xi, k))^2$; (17)

when $a = 1 - k^2, b = 2(1 + k^2)$,, and $c = 1 - k^2$, then $z(\xi) = (nc(\xi, k) \pm sc(\xi, k))^2$; (18)

when $a = k^4, b = 2(-2 + k^2)$, and $c = 1$, then $z(\xi) = (ns(\xi, k) \pm ds(\xi, k))^2$; (19)

when $a = k^2, b = 2(-2 + k^2)$, and $c = k^2$, then $z(\xi) = (sn(\xi, k) \pm icn(\xi, k))^2$,

and $z(\xi) = \frac{dn^2(\xi,k)}{(\sqrt{1-k^2}sn(\xi,k) \pm cn(\xi,k))^2}$; (20)

when $a = 1, b = -2(-1 + 2k^2)$, and $c = 1$, then $z(\xi) = (ksn(\xi, k) \pm idn(\xi, k))^2$; (21)

when $a = 1 - k^2, b = 2(1 + k^2)$, and $c = 1 - k^2$, then $z(\xi) = \frac{cn^2(\xi,k)}{(1 \pm sn(\xi,k))^2}$; (22)

when $a = -1 + k^2, b = 2(1 + k^2)$, and $c = -1 + k^2$, then $z(\xi) = \frac{dn^2(\xi,k)}{(1 \pm ksn(\xi,k))^2}$; (23)

when $a = k^2, b = 2(-2 + k^2)$, and $c = k^2$, then $z(\xi) = \frac{k^2 sn^2(\xi,k)}{(1 \pm dn(\xi,k))^2}$; (24)

when $a = 1, b = -2(-1 + 2k^2)$, and $c = 1$, then $z(\xi) = \frac{sn^2(\xi,k)}{(1 \pm cn(\xi,k))^2}$; (25)

when $a = 1, b = 2(-2 + k^2)$, and $c = k^4$, then $z(\xi) = \frac{sn^2(\xi,k)}{(1 \pm dn(\xi,k))^2}$ (26)

Substituting the expression (2) and Eq. (3) into Eq. (1), and setting the coefficients of $z^j(\xi)(j = 0,1,2,\ldots.)$ and $x^s z^i \sqrt{az(\xi) + bz^2(\xi) + cz^3(\xi)}$ $(s = 0,1; i = 0,1,\ldots.)$ to zero yield an over-determined partial differential equation set about $a, b, g_0(t), g_1(t), p(t), q(t)$. We obtain the solutions of the equation set with the help of Mathematica. And then substituting each of solutions of the equation set, *i.e.* each of the expressions (4)- (26) into the expression (2) respectively, we obbtain the Jacobi elliptic function-like solutions, the soliton-like solutions and the triangle function wave solution to the nonlinear evolution equation (1).

3. The first kind of KdV equation with the variable coefficients

$$u_t + \alpha(t)uu_x + \beta(t)u_{xxx} = 0. \tag{27}$$

In Eq. (27), assume that

$$u(x,t) = g_0(t) + g_1(t)z(p(t)x + q(t)). \tag{28}$$

Substituting Eq. (3) and the expression (28) into Eq. (27), and setting the coefficients of $z^j(\xi)(j = 0,1)$, and $x^s z^i(\xi)\sqrt{az(\xi) + bz^2(\xi) + cz^3(\xi)}$ $(s = 0,1; i = 0,1)$ to zero yields the following over-determined partial differential equation set.

$$g_0'(t) = 0,$$

$$g_1'(t) = 0,$$

$$g_1(t)p'(t) = 0,$$

$$g_1^2(t)p(t)\alpha(t) + 3cg_1(t)p^3(t)\beta(t) = 0,$$

$$g_1(t)g_0(t)p(t)\alpha(t) + bg_1(t)p^3(t)\beta(t) + g_1(t)q'(t) = 0,$$

With the aid of Mathematica, we have the following solutions of the equation set:

$$g_0(t) = g_0, g_1(t) = g_1, p(t) = p, \alpha(t) = \frac{-3cp^2\beta(t)}{g_1}, \qquad q(t) = -\int(g_0 p\alpha(t) + bp^3\beta(t))dt,$$

where g_0, g_1, p, b and c are constants, $g_0 \neq 0$, and $g_1 \neq 0$.

Substituting the expressions (29) together with the expressions (4) – (26) into the expression (28) respectively, we obtain Jacobi ellipti function-like exact solution of Eq (27) as follows:

$u_{1,1}(x,t) = g_0 + g_1 sn^2(\xi,k);$ $u_{2,1}(x,t) = g_0 + g_1 cn^2(\xi,k);$

$u_{3,1}(x,t) = g_0 + g_1 dn^2(\xi,k);$ $u_{4,1}(x,t) = g_0 + g_1 ns^2(\xi,k);$

$u_{5,1}(x,t) = g_0 + g_1 nc^2(\xi,k);$ $u_{6,1}(x,t) = g_0 + g_1 nd^2(\xi,k);$

$u_{7,1}(x,t) = g_0 + g_1 sc^2(\xi,k);$ $u_{8,1}(x,t) = g_0 + g_1 sd^2(\xi,k);$

$u_{9,1}(x,t) = g_0 + g_1 cs^2(\xi,k);$ $u_{10,1}(x,t) = g_0 + g_1 cd^2(\xi,k);$

$u_{11,1}(x,t) = g_0 + g_1 ds^2(\xi,k);$ $u_{12,1}(x,t) = g_0 + g_1 dc^2(\xi,k);$

$u_{13,1}(x,t) = g_0 + g_1(kcn(\xi,k) \pm dn(\xi,k))^2;$ $u_{14,1}(x,t) = g_0 + g_1(ns(\xi,k) \pm cs(\xi,k))^2;$

$u_{15,1}(x,t) = g_0 + g_1(nc(\xi,k) \pm sc(\xi,k))^2;$ $u_{16,1}(x,t) = g_0 + g_1(ns(\xi,k) \pm ds(\xi,k))^2;$

$u_{17,1}(x,t) = g_0 + g_1(sn(\xi,k) \pm icn(\xi,k))^2;$ $u_{18,1}(x,t) = g_0 + \frac{g_1 dn^2(\xi,k)}{(\sqrt{1-k^2}sn(\xi,k) \pm cn(\xi,k))^2};$

$u_{19,1}(x,t) = g_0 + g_1(ksn(\xi,k) \pm idn(\xi,k))^2;$ $u_{20,1}(x,t) = g_0 + \frac{g_1 cn^2(\xi,k)}{(1 \pm sn(\xi,k))^2};$

$u_{21,1}(x,t) = g_0 + \frac{g_1 dn^2(\xi,k)}{(1 \pm ksn(\xi,k))^2};$ $u_{22,1}(x,t) = g_0 + \frac{g_1 k^2 sn^2(\xi,k)}{(1 \pm dn(\xi,k))^2};$

$u_{23,1}(x,t) = g_0 + \frac{g_1 sn^2(\xi,k)}{(1 \pm cn(\xi,k))^2};$ $u_{24,1}(x,t) = g_0 + \frac{g_1 sn^2(\xi,k)}{(1 \pm dn(\xi,k))^2};$

where $\xi = px - \int (g_0 p\alpha(t) + bp^3\beta(t))dt, p, b, c, g_1$ are constants, $g_0 \neq 0$, and $g_1 \neq 0$,

The Jacobi elliptic functio-like exact solutions above degenerate into the following soliton-like solutions. where $k = 1$;

$$u_{1,2}(x,t) = g_0 + g_1\, tanh^2(px - \int (g_0 p\alpha(t) - 8p^3\beta(t)dt), \left(\alpha(t) = \frac{-12p^2\beta(t)}{g_1}\right);$$

$$u_{2,2}(x,t) = g_0 + g_1 \operatorname{sech}^2\left(px - \int (g_0 p\alpha(t) - 4p^3\beta(t)dt\right), \left(\alpha(t) = \frac{12p^2\beta(t)}{g_1}\right);$$

$$u_{3,2}(x,t) = g_0 + g_1 \coth^2\left(px - \int (g_0 p\alpha(t) - 8p^3\beta(t)dt\right), \left(\alpha(t) = \frac{-12p^2\beta(t)}{g_1}\right);$$

$$u_{4,2}(x,t) = g_0 + \frac{g_1\, sech^2(px - \int (g_0 p\alpha(t) + 4p^3\beta(t)dt)}{tanh^2(px - \int (g_0 p\alpha(t) + 4p^3\beta(t)dt))}, \left(\alpha(t) = \frac{-12p^2\beta(t)}{g_1}\right);$$

$$u_{5,2}(x,t) = g_0 + 4g_1\, sech^2(px - \int (g_0 p\alpha(t) + 4p^3\beta(t))dt), \left(\alpha(t) = \frac{3p^2\beta(t)}{g_1}\right);$$

$$u_{6,2}(x,t) = g_0 + \frac{g_1(1 \pm sech(px - \int (g_0 p\alpha(t) - 2p^3\beta(t))dt))^2}{tanh^2(px - \int (g_0 p\alpha(t) - 2p^3\beta(t))dt)}, \left(\alpha(t) = \frac{-3p^2\beta(t)}{g_1}\right)$$

$$u_{7,2}(x,t) = g_0 + g_1(tanh(px - \int (g_0 p\alpha(t) - 2p^3\beta(t))\, dt)$$
$$\pm i\, sech(px - \int (g_0 p\alpha(t) - 2p^3\beta(t)dt))^2, \left(\alpha(t) = \frac{-3p^2\beta(t)}{g_1}\right);$$

$$u_{8,2}(x,t) = g_0 + \frac{g_1 \tanh^2\left(px - \int (g_0 p\alpha(t) - 2p^3\beta(t))dt\right)}{\left(1 \pm \operatorname{sech}\left(px - \int (g_0 p\alpha(t) - 2p^3\beta(t)dt\right)\right)^2}, \left(\alpha(t) = \frac{-3p^2\beta(t)}{g_1}\right)$$

The Jacobi ellipti function-like exact solutions above degenerate into the following triangle function solutions. where $k = 0$

$$u_{1,3}(x,t) = g_0 + g_1\, csc^2(px - \int (g_0 p\alpha(t) - 4p^3\beta(t)dt), \left(\alpha(t) = \frac{-12p^2\beta(t)}{g_1}\right)$$

$$u_{2,3}(x,t) = g_0 + g_1\, sec^2(px - \int (g_0 p\alpha(t) - 4p^3\beta(t)dt), \left(\alpha(t) = \frac{-12p^2\beta(t)}{g_1}\right)$$

$$u_{3,3}(x,t) = g_0 + g_1\, tan^2(px - \int (g_0 p\alpha(t) - 8p^3\beta(t)dt), \left(\alpha(t) = \frac{-12p^2\beta(t)}{g_1}\right)$$

$$u_{4,3}(x,t) = g_0 + g_1 \cot^2(px - \int(g_0 p\alpha(t) - 8p^3\beta(t)dt), \left(\alpha(t) = \frac{-12p^2\beta(t)}{g_1}\right)$$

$$u_{5,3}(x,t) = g_0 + g_1(\csc(px - \int(g_0 p\alpha(t) + 2p^3\beta(t))dt)) \pm (px - \int(g_0 p\alpha(t) + 2p^3\beta(t)dt))^3,$$

$$(\alpha(t) = \frac{-3p^2\beta(t)}{g_1})$$

$$u_{6,3}(x,t) = g_0 + g_1\left(\sec(px - \int(g_0 p\alpha(t) + 2p^3\beta(t))dt\right) \pm \left(px - \int(g_0 p\alpha(t) + 2p^3\beta(t))dt\right)$$

$$\left(\alpha(t) = \frac{-3p^2\beta(t)}{g_1}\right)$$

$$u_{7,3}(x,t) = g_0 + 4g_1 \csc^2(px - \int(g_0 p\alpha(t) - 4p^3\beta(t))dt), \left(\alpha(t) = \frac{-3p^2\beta(t)}{g_1}\right)$$

$$u_{8,3}(x,t) = g_0 + \frac{g_1 \cos^2(px - \int(g_0 p\alpha(t) + 2p^3\beta(t))dt)}{(1 \pm \sin(px - \int(g_0 p\alpha(t) + 2p^3\beta(t)dt))^2}, \left(\alpha(t) = \frac{-3p^2\beta(t)}{g_1}\right)$$

$$u_{9,3}(x,t) = g_0 + \frac{g_1 \sin^2(px - \int(g_0 p\alpha(t) + 2p^3\beta(t))dt)}{(1 \pm \cos(px - \int(g_0 p\alpha(t) + 2p^3\beta(t)dt))^2}, \left(\alpha(t) = \frac{-3p^2\beta(t)}{g_1}\right)$$

4. Discussion and Conclusions :

In this chapter, we introduce a method using an auxiliary equation with a function transformation to find new solutions for first kind KdV equation with variable coefficients. These solutions include exact solutions using Jacobi elliptic functions, degenerate soliton-like solutions, and triangular function wave solutions. The specific solutions are:

$u_{1,1}(x,t) \sim u_{2,1}(x,t)$,

$u_{1,2}(x,t) \sim u_{2,2}(x,t), u_{3,2}(x,t), u_{5,2}(x,t), u_{2,3}(x,t)$;

$u_{1,4}(x,t) \sim u_{2,4}, u_{1,5}(x,t) \sim u_{2,5}(x,t), u_{3,5}(x,t), u_{5,5}(x,t), u_{2,6}(x,t)$;

$u_{1,7}(x,t) \sim u_{2,7}(x,t), u_{5,8}(x,t), u_{2,9}(x,t)$.

$u_{1,8}(x,t) \sim u_{2,8}(x,t), u_{3,8}(x,t)$,

Additionally, we have discovered some new solutions in this paper. This new method is important for constructing exact solutions using Jacobi elliptic functions for nonlinear evolution equations with variable coefficients.

5. References

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