Modelling the effect of disease on plant-herbivore dynamics with special reference to orange trees

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Abstract

The mathematical model is considered here to investigate the effects of pathogen on the orange trees in the presence of Herbivores. It is assumed in the model that the orange trees are directly infected by pathogen, and adversely affected by herbivores. The local and global stability analysis of all the equilibrium points of the mathematical model are discussed. Through the analysis it has been derived that the density of orange trees reduces in the presence of pathogen.

Keywords: Plants, Herbivore, Plants diseases, Stability.

1 Introduction

The rising number of invasive and aggressive pathogenic fungus is one of the biggest risks to sustainable agricultural production and food security. They significantly reduce crop output and quality [1, 2]. Citrus species are known to be infected by a number of Phyllosticta species, which cause a variety of disease signs, including leaf and fruit spots. P. citricarpa, which causes citrus black spot, a foliar and fruit disease,

is one of the most significant species [3]. The most damaging Fungal citrus disease is the citrus black spot caused by the fungus Phylosticta citricarpa which causes the loss of yield in different countries of the world like Brazil, Australia etc.
The disease is worldwide distributed affecting all varieties of citrus within tropical and sub-tropical citrus production regions, particularly in warm and humid climates. The fungus forms latent infection in citrus tissues thereby causing disease and reducing the yield [4–6]. Also another pathogen namely C. canker, one of the most devastating biotic stressors to citrus, has a significant economic impact on the sector, limiting trade and output. It affects all commercial citrus types as well as a wide range of associated rutaceous species. In Northeast and Northwest Argentina, Xanthomonas infection is considered an endemic disease, affecting up to 10% of commercial citrus orchards [7, 8]. The presence of numerous diseases and pests in fruit, as well as harvesting fruit at varying stages of maturity, are negative variables that limit fruit marketability, reduce economic value and increase fruit waste [9]. The combined impacts of herbivores and pathogenic fungus on tree growth are mediated by differences in tree species and other plant characteristics [10, 11]. The composition and operation of plant communities are primarily influenced by herbivores and fungal diseases. The diversity and functional traits of their host plants operate as a conduit for the effects of herbivores and diseases. However, the combined effects of herbivory and plant pathogen damages and their implications is yet to be addressed mathematically [12]. The disease can be controlled by the virtue of fungicide decreasing outbreaks, however, doing so, increases cost of output and may also damage ecosystem [14]. The scientists [15] have developed a mathematical model for biological management and tactics. They took into account three populations of orange trees, as well as disease and beneficial fungi, and observed transcritical bifurcation in the model's ultimate behavior. Both farmers and applied ecologists have been notified of these findings.

In view of the research article [15] we have investigated the interaction dynamics between orange trees and herbivores in the presence of pathogenic fungus induced infectious disease. The goal of this work is to investigate and assess the effects of disease on plant-herbivore dynamics using a mathematical model. We discuss the behavior of a mathematical model consists of orange trees, Infected orange trees, Herbivores and Pathogenic fungus. The theoretical results are supported by numerical results, suggesting potential management tactics [15].

2 Model

In the proposed model, the interaction among the Orange tree, Infected orange tree, Herbivore and the Pathogenic fungus is being studied and analyzed.
In view of the above assumptions, let T_1 denotes the density of Orange tree, T_2 denotes the density of Infected orange trees, H denotes the Herbivore density, P denotes the density of Pathogenic fungus. The following set of equations demonstrate the mathematical model as:

$$\frac{dT_1}{dt} = \Lambda - d_1 T_1 - a_1 T_1 P - a_2 T_1 H - \beta_1 T_1^2 \tag{1}$$

$$\frac{dT_2}{dt} = a_1T_1P - d_2T_2 - e_1T_2H - \beta_2T_2^2 \tag{2}$$

$$\frac{dH}{dt} = k_1a_2T_1H - d_3H - k_2e_1T_2H - \beta_3H^2 \tag{3}$$

$$\frac{dP}{dt} = (\frac{\mu_{max}\ s}{k_s + s})P + k_3d_2T_2 - c_1P + k_4d_3H \tag{4}$$

and the initial conditions are $T_1(0) > 0,\ T_2(0) > 0,\ H(0) > 0$ and $P(0) > 0$.

Monod equation: Jacques Monod created the monod equation in the 1940, and the equation is $m = \frac{\mu_{max}s}{k_s+s}$ where μ is the specific growth rate(1/time), μ is the culture's maximal specific growth rate (1/time), s is the substrate concentration (mass/volume) required for the growth of Pathogenic fungus and k_s is the half-saturation constant, or affinity constant (mass/volume). Here, s is considered to be constant [15].

3 Table

Table 1: Parameters with their biological understandings / meanings.

Parameters	**Biological meanings**
Λ constant	Reproduction rate of orange trees
d_1	Natural death rate of T_1
a_1	Infection rate
a_2	The maximum value at which per capita reduction rate of orange trees can attain due to H because of grazing
d_2	Death rate of T_2
e_1	The maximum value at which per capita reduction rate of T_2 can occur due to H because of grazing.
k_1	Conversion rate
d_3	Natural death rate of H
k_2	Death rate due to disease on account of eating of infected trees.
k_3	Decomposition rate
c_1	Natural death rate
k_4	Decomposition coefficient
β_1	Crowding effect of T_1
β_2	Crowding effect of T_2
β_3	Crowding effect of H
$m = \frac{\mu_{max}\ s}{k_s+s}$	Monod type of growth rate function

4 Boundedness of the model

Lemma 1 All the solutions of the model (1) - (4) will lie in the region

$$\Omega = \left\{(T_1, T_2, H, P) \in R_+^4 : 0 \leq k_1T_1(t) + k_2T_2(t) + H(t) + P(t) \leq \frac{k_1\Lambda}{\theta_1}\right\}$$

where $\theta_1 = min\left\{(d_1k_1, d_2(k_2 - k_3), d_3(1 - k_4), \frac{c_1(k_s+s)-\mu_{max}s}{k_s+s})\right\}, k_2 > k_3, 1 > k_4,$ $c_1(k_s + s) > \mu_{max}s$.

Proof: Now we consider a time dependent function:

$$W(t) = k_1T_1(t) + k_2T_2(t) + H(t) + P(t)$$

by using (1)-(4), we get

$$\frac{dW(t)}{dt} \leq \Lambda k_1 - d_1k_1T_1 - d_2(k_2 - k_3)T_2 - d_3(1 - k_4)H - \left(\frac{c_1(k_s + s) - \mu_{max}s}{k_s + s}\right) P$$

$$\frac{dW(t)}{dt} \leq \Lambda k_1 - \theta(W)$$

where $\theta_1 = min\left\{(d_1k_1), d_2(k_2 - k_3), d_3(1 - k_4), \left(\frac{c_1(k_s+s)-\mu_{max}s}{k_s+s}\right)\right\}$, $k_2 > k_3$, $1 > k_4$, $k_2 << k_1$, $c_1(k_s + s) > \mu_{max}s$.
Now, on applying the theorem on differential inequalities, we obtain
$0 < W(t) \leq W(0)e^{-\theta t} + \frac{\Lambda k_1}{\theta}$ as $t \to \infty$ then we get

$$0 \leq W(t) \leq \frac{k_1\Lambda}{\theta}$$

Hence, all the solutions of the model are bounded in Ω.

5 Equilibria of the model

In this section, we discuss the existence of all possible equilibrium points of the model (1)-(4). The model has at most two equilibrium points, namely $E_1(\overline{T_1}, \overline{T_2}, 0, \overline{P})$ and $E_2(\tilde{T_1}, \tilde{T_2}, \tilde{H}, \tilde{P})$, which are discussed as follows:
Existence of E_1: The first equilibrium point $E_1(\overline{T_1}, \overline{T_2}, 0, \overline{P})$ which is also known as boundary equilibrium point.
From (4), we get,

$$P = \frac{k_3d_2T_2}{c_1 - m} \tag{5}$$

$P > 0$ if $c_1 > m$, and put the value of (5) in (2), we get

$$\overline{T_2} = \frac{1}{\beta_2}\left(\frac{a_1k_3d_2\overline{T_1}}{c_1 - m} - d_2\right), \quad \frac{a_1k_3d_2\overline{T_1}}{c_1 - m} > d_2, \qquad c_1 > m \tag{6}$$

and put the value of (6) in (1), we get

$$\left(\frac{a_1^2k_3^2d_2^2}{\beta_2(c_1 - m)} + \beta_2\right)\overline{T_1}^2 + \left(d_1 - \frac{a_1k_3d_2^2c_1}{\beta_2(c_1 - m)} + \frac{a_1d_2m}{\beta_2(c_1 - m)}\right)\overline{T_1} - \Lambda = 0$$

$$\overline{T_1} = \frac{-\pi_1 + \sqrt{(\pi_1)^2 + 4\Lambda\left(\frac{a_1^2k_3^2d_2^2}{\beta_2(c_1-m)} + \beta_1\right)}}{2\left[\frac{a_1^2k_3^2d_2^2}{\beta_2(c_1-m)} + \beta_1\right]} \tag{7}$$

$$\overline{P} = \frac{k_3^2 d_2^2 a_1 \overline{T_1}}{\beta_2(c_1 - m)^2} - \frac{k_3 d_2^2}{\beta_2(c_1 - m)} = \frac{k_3 d_2^2}{\beta_2(c_1 - m)} \left(\frac{k_3 a_1 \overline{T_1}}{c_1 - m} - 1 \right)$$

if,

$$\frac{k_3 a_1 \overline{T_1}}{c_1 - m} > 1, \; k_3 a_1 \overline{T_1} > (c_1 - m), \; k_3 a_1 \overline{T_1} + m > c_1$$

where,

$$\pi_1 = d_1 - \frac{a_1 k_3 d_2^2 c_1}{\beta_2(c_1 - m)^2} + \frac{a_1 d_2^2 m k_3}{\beta_2(c_1 - m)}.$$

Existence of E_2: Now we will show the existence of the point E_2, by the formation of two isoclines $F_1(\tilde{T_1}, \tilde{T_2})$ and $F_2(\tilde{T_1}, \tilde{T_2})$ as follows:
From (3)

$$\tilde{H} = \frac{1}{\beta_3}[k_1 a_2 \tilde{T_1} - d_3 + k_2 e_1 \tilde{T_2}], \; (k_1 a_2 \tilde{T_1} + k_2 e_1 \tilde{T_2}) > d_3 \tag{8}$$

from (1), (2) and (8), we get
$F_1(\tilde{T_1}, \tilde{T_2}) = \beta_3 \Lambda - \beta_3 d_1 \tilde{T_1} - \beta_1 \beta_3 \tilde{T_1}^2 - d_2 \beta_3 \tilde{T_2} - \beta_2 \beta_3 \tilde{T_2}^2 - a_2^2 k_1 \tilde{T_1}^2 + a_2 d_3 \tilde{T_1}$
$- a_2 k_2 e_1 \tilde{T_1} \tilde{T_2} - a_2 k_1 e_1 \tilde{T_1} \tilde{T_2} + e_1 d_3 \tilde{T_2} - k_2 e_1^2 \tilde{T_2}^2$
Also from (4)

$$\tilde{P} = \frac{k_3 d_2 \tilde{T_2} + k_4 d_3 \tilde{H}}{c_1 - m}, \; c_1 > m \tag{9}$$

Now putting the value of (8), (9) in (2), we get
$F_2(\tilde{T_1}, \tilde{T_2}) = \beta_3 a_1 d_2 k_3 \tilde{T_1} \tilde{T_2} + (a_1 d_3 k_4 \tilde{T_1})(k_1 a_2 \tilde{T_1} - d_3 + e_1 k_2 \tilde{T_2}) - \beta_3 d_2 \tilde{T_2}(c_1 - m)$
$- e_1 \tilde{T_2}(c_1 - m)(k_1 a_2 \tilde{T_1} - d_3 + k_2 e_1 \tilde{T_2}) - \beta_2 \beta_3 \tilde{T_2}^2 (c_1 - m).$
Now in order to show that $\tilde{T_1}$ and $\tilde{T_2}$ exist, the two isoclines $F_1(\tilde{T_1}, \tilde{T_2}) = 0$ and $F_2(\tilde{T_1}, \tilde{T_2}) = 0$ must intersect, now we note that $F_2(0, 0) = 0$.
$\implies$ $F_2(\tilde{T_1}, \tilde{T_2})$ passes through the origin and now we will show that $F_2(\tilde{T_1}, \tilde{T_2})$ is monotonically increasing as follows, now

$$\frac{d\tilde{T_1}}{d\tilde{T_2}} = \frac{\beta_3 d_2 \phi + e_1 \phi \chi + 2\beta_2 \beta_3 \tilde{T_2} \phi - \beta_3 a_1 k_3 d_2 \tilde{T_1} - a_1 k_4 d_3 k_2 e_1 \tilde{T_2}}{\beta_3 a_1 k_3 d_2 \tilde{T_2} + 2 a_1 a_2 d_3 k_4 k_1 \tilde{T_1} + a_1 d_3 e_1 k_2 k_4 \tilde{T_2} - a_1 d_3^2 k_4}$$

where, $\chi = k_1 a_2 \tilde{T_1} + 2 k_2 e_1 \tilde{T_2} - d_3$, $\phi = c_1 - m$.
$\implies \frac{d\tilde{T_1}}{d\tilde{T_2}} > 0$ if the following condition are satisfied, *i.e.*,

1. $\beta_3 d_2 (c_1 - m) + e_1 (c_1 - m)(k_1 a_2 \tilde{T_1}) + e_1 (c_1 - m)(2 k_2 e_1 \tilde{T_2}) + 2\beta_2 \beta_3 \tilde{T_2}(c_1 - m) > e_1 (c_1 - m)(-d_3) + \beta_3 a_1 k_3 d_2 \tilde{T_1} + a_1 k_4 d_3 k_2 e_1 \tilde{T_2}$
2. $\beta_3 a_1 k_3 d_2 \tilde{T_2} + 2 a_1 a_2 d_3 k_4 k_1 \tilde{T_1} + a_1 d_3 e_1 k_2 k_4 \tilde{T_2} > a_1 d_3^2 k_4$

Hence $F_2(\tilde{T_1}, \tilde{T_2})$ is monotonically increasing and passing through origin, again for the Isocline $F_1(\tilde{T_1}, \tilde{T_2})$, we have $F_1(0, \tilde{T_2}) = 0$

$$\tilde{T_2} = \frac{-(d_2 \beta_3 - e_1 d_3) + \sqrt{(d_2 \beta_3 - e_1 d_3)^2 + 4\beta_2 \beta_3^2 \Lambda + 4\beta_3 k_2 e_1^2 \Lambda}}{2(\beta_2 \beta_3 + k_2 e_1^2)} = \psi_1(Say)$$

and $F_1(\tilde{T}_1, 0) = 0$

$$\tilde{T}_1 = \frac{-(\beta_3 d_1 - a_2 d_3) + \sqrt{(\beta_3 d_1 - a_2 d_3)^2 + 4\beta_1\beta_3^2\Lambda + 4\beta_3 a_2^2 k_1\Lambda}}{2(\beta_1\beta_3 + a_2^2 k_1)} = \psi_2(Say)$$

Also

$$\frac{d\tilde{T}_1}{d\tilde{T}_2} = -\left(\frac{d_2\beta_3 + 2\beta_2\beta_3 + a_2k_2e_1\tilde{T}_1 + e_1k_1a_2\tilde{T}_1 + 2k_2e_1^2\tilde{T}_2 - e_1d_3}{\beta_3d_1 + 2\beta_1\beta_3\tilde{T}_1 + 2a_2^2k_1\tilde{T}_1 + a_2k_2e_1\tilde{T}_2 + e_1k_1a_2\tilde{T}_2 - a_2d_3}\right) < 0.$$

Hence $F_1(\tilde{T}_1, \tilde{T}_2)$ and $F_2(\tilde{T}_1, \tilde{T}_2)$ will intersect in the positive plane having the cordinates for the isocline $F_1(\tilde{T}_1, \tilde{T}_2)$ as $(0, \tilde{T}_2)$ and $(\tilde{T}_1, 0)$. Therefore the intersect equilibrium point is shown below by the intersection of $F_1(\tilde{T}_1, \tilde{T}_2)$ and $F_2(\tilde{T}_1, \tilde{T}_2)$.

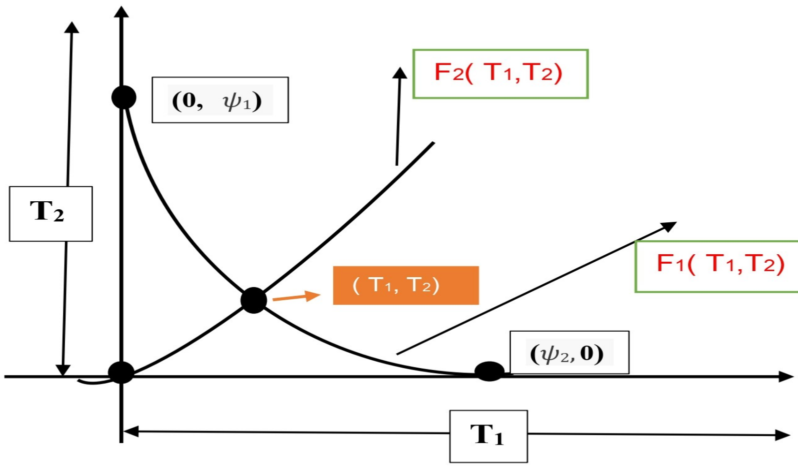

Figure 1: Isocline graph for equilibrium point E_2

6 Local stability of Model

6.1 Local stability for E_1. The variational matrix about E_1 is:

$$J_1 = \begin{bmatrix} -(d_1 + a_1\overline{P}a_2H) & 0 & a_2\overline{T_1} & -a_1\overline{T_1} \\ a_1\overline{P} & -(d_2 + 2\beta_2\overline{T_2}) & -e_1\overline{T_2} & a_1\overline{T_1} \\ 0 & 0 & (k_1a_2\overline{T_1} - d_3 + k_2e_1\overline{T_2}) & 0 \\ 0 & k_3d_2 & k_4d_3 & m - c_1 \end{bmatrix}$$

The characteristic equation of the above matrix is:

$$(\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3)(-\lambda - (d_3 - k_1a_2\overline{T_1} - k_2e_1\overline{T_2})) = 0$$

one of the eigenvalues of the characteristic equation is $\lambda_1 = -(d_3 - k_1 a_2 \overline{T_1} - k_2 e_1 \overline{T_2})$, and the other three eigenvalues are given by the following equation:

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0 \tag{10}$$

where,

$$\begin{aligned}
A_1 &= (c_1 + d_2 + 2\beta_2\overline{T_2} + d_1 + a_1\overline{P} + a_2\overline{H} - m),\\
A_2 &= (2\beta_2 c_1\overline{T_2} + c_1 d_1 + c_1 a_1\overline{P} + c_1 a_2\overline{H} + d_1 d_2 + a_1 d_2\overline{P} + a_2 d_2\overline{H} + 2\beta_2 d_1\overline{T_2}\\
&\quad +2\beta_2 a_1\overline{PT_2} + 2\beta_2 a_2\overline{HT_2} - k_3 d_2 a_1\overline{T_1} + c_1 d_2 - a_2 m\overline{H} - a_1 m\overline{P} - d_1 m\\
&\quad -2\beta_2 m\overline{T_2} - d_2 m),\\
A_3 &= (c_1 d_1 d_2 + a_1 c_1 d_2\overline{P} + a_2 c_1 d_2\overline{H} + 2\beta_2 c_1 d_1\overline{T_2} + 2\beta_2 c_1 a_1\overline{PT_2} + 2\beta_2 c_1 a_2\overline{HT_2}\\
&\quad -d_1 d_2 m - a_1 d_2 m\overline{P} - a_2 d_2 m\overline{H} - 2\beta_2 d_1 m\overline{T_2} - 2\beta_2 m a_1\overline{T_2 P}\\
&\quad -2\beta_2 m a_2\overline{T_2 H} - k_3 d_2 a_1 d_1\overline{T_1} - k_3 d_2 a_1 a_2\overline{T_1 H}).
\end{aligned}$$

According to Routh Hurwitz Criteria, E_1 is locally stable if $A_1 > 0; A_2 > 0; A_3 > 0$ and $A_1A_2 > A_3$ hold. From these expressions it is difficult to interpret the results in the ecological terms, although all these conditions are numerically established by considering a set of different parametric values.

6.2 Local stability for E_2:

Theorem 1: In the region Ω, if the following conditions hold:

$$d_2 + e_1\overline{H} > 2\beta_2\overline{T_2} \tag{11}$$

$$d_3 + 2\beta_3\overline{H} > k_2 e_1\overline{T_2} + k_1 a_2\overline{T_1} \tag{12}$$

$$c_1 > m \tag{13}$$

$$J_2 > \frac{\overline{T_1}}{k_1\overline{H}} \tag{14}$$

$$J_1 < \frac{k_2\overline{H}J_2}{\overline{T_2}} \tag{15}$$

where,

$$J_1 < \frac{\gamma(d_2 + e_1\overline{H} - 2\beta_2\overline{T_2})}{9a_1^2\overline{P}^2} \tag{16}$$

$$J_2 < \frac{(3k_4 d_3)^2(3a_1\overline{T_1})^2}{\gamma(d_3 + 2\beta_3\overline{H} - k_2 e_1\overline{T_2} - a_2 k_1\overline{T_1})(c_1 - m)^2} \tag{17}$$

$$J_3 > \frac{(3a_1\overline{T_1})^2}{\gamma(c_1 - m)} \tag{18}$$

where, $\gamma = (d_1 + a_1\overline{P} + a_2\overline{H})$, then $E_2(\overline{T_1}, \overline{T_2}, \overline{H}, \overline{P})$ will be locally asymptotically stable.

Proof: We first linearize the model about the equilibrium E_2 by using the following transformation:

$$T_1 = \overline{T_1} + n_1$$
$$T_1 = \overline{T_2} + n_2$$
$$H = \overline{H} + n_3$$
$$P = \overline{P} + n_4$$

where, n_1, n_2, n_3, n_4 are small perturbation around E_2. Then we get the following linearized the model,

$$\frac{dn_1}{dt} = -(d_1 + a_1\overline{P} + a_2\overline{H} + 2\beta_1\overline{T_1})n_1 - a_2\overline{T_1}n_3 - a_1\overline{T_1}n_4$$
$$\frac{dn_2}{dt} = a_1\overline{P}n_1 - (d_2 + e_1\overline{H} - \beta_2 n_2 - 2\beta_2\overline{T_2})n_2 - e_1\overline{T_2}n_3 + a_1\overline{T_1}n_4$$
$$\frac{dn_3}{dt} = k_1 a_2\overline{H}n_1 + k_2 e_1\overline{H}n_2 + (k_2 e_1\overline{T_2} + k_1 a_2\overline{T_1} - d_3 - 2\beta_3\overline{H})n_3$$
$$\frac{dn_4}{dt} = k_3 d_2 n_2 + k_4 d_3 n_3 + (m - c_1)n_4$$

Now consider the following positive definite function:

$$V = \frac{n_1^2}{2} + \frac{J_1 n_2^2}{2} + \frac{J_2 n_3^2}{2} + \frac{J_3 n_4^2}{2}$$
$$\frac{dV}{dt} = n_1\frac{dn_1}{dt} + J_1 n_2\frac{dn_2}{dt} + J_2 n_3\frac{dn_3}{dt} + J_3 n_4\frac{dn_4}{dt}$$

$$\begin{aligned}\frac{dV}{dt} &= -[(d_1 + a_1\overline{P} + a_2\overline{H})n_1^2 + (J_1 d_2 + J_1 e_1\overline{H} - 2\beta_2 J_1\overline{T_2})n_2^2 \\ &+(J_2 d_3 + 2J_2\beta_3\overline{H} - J_2 k_2 e_1\overline{T_2} - J_2 k_1 a_2\overline{T_1})n_3^2 + (J_3 c_1 - J_3 m)n_4^2 \\ &-(J_1 a_1\overline{P})n_1 n_2 - (k_1 a_2 J_2\overline{H} - a_2\overline{T_1})n_1 n_3 + (a_1\overline{T_1})n_1 n_4 \\ &-(J_2 k_2 e_1\overline{H} - J_1 e_1\overline{T_2})n_2 n_3 - (J_3 k_3 d_2 + J_1 a_1\overline{T_1})n_2 n_4 - (J_3 k_4 d_3)n_3 n_4]\end{aligned}$$

Now using the sylvester's criterion in the quadratic forms:

$$\begin{aligned}\frac{dV}{dt} \leq -[&\left(b_{11}\frac{n_1^2}{2} - b_{12}n_1 n_2 + b_{22}\frac{n_2^2}{2}\right) + \left(b_{11}\frac{n_1^2}{2} - b_{13}n_1 n_3 + b_{33}\frac{n_3^2}{2}\right) \\ &+ \left(b_{11}\frac{n_1^2}{2} - b_{14}n_1 n_4 + b_{44}\frac{n_4^2}{2}\right) + \left(b_{22}\frac{n_2^2}{2} - b_{23}n_2 n_3 + b_{33}\frac{n_3^2}{2}\right) \\ &+ \left(b_{22}\frac{n_1^2}{2} - b_{24}n_2 n_4 + b_{44}\frac{n_4^2}{2}\right) + \left(b_{33}\frac{n_1^2}{2} - b_{34}n_3 n_4 + b_{22}\frac{n_4^2}{2}\right)]\end{aligned}$$

Where,

$$b_{11}=\frac{(d_1+a_1\overline{P}+a_2\overline{H})}{3}, b_{22}=\frac{(J_1d_2+J_1e_1\overline{H}-2\beta_2J_1\overline{T_2})}{3}, b_{23}=(J_2k_2e_1\overline{H}-J_1e_1\overline{T_2}),$$

$$b_{33}=\frac{(J_2d_3+2J_2\beta_3\overline{H}-J_2k_2e_1\overline{T_2}-J_2k_1a_2\overline{T_1})}{3}, b_{44}=\frac{(J_3c_1-J_3m)}{3}, b_{12}=(J_1a_1\overline{P}),$$

$$b_{13}=(k_1a_2J_2\overline{H}-a_2\overline{T_1}), b_{14}=(a_1\overline{T_1}), b_{24}=(J_3k_3d_2+J_1a_1\overline{T_1}), b_{34}=(J_3k_4d_3).$$

Sufficient conditions for $\frac{dV}{dt}$ to be negative definite are that the following inequalities hold:

$$b_{22}>0 \tag{19}$$
$$b_{33}>0 \tag{20}$$
$$b_{44}>0 \tag{21}$$
$$b_{13}>0 \tag{22}$$
$$b_{23}>0 \tag{23}$$
$$b_{11}b_{12}>b_{12}^2 \tag{24}$$
$$b_{33}b_{44}>b_{34}^2 \tag{25}$$
$$b_{11}b_{44}>b_{14}^2 \tag{26}$$

We note that the inequalities, $(11)\implies(19),(12)\implies(20),(13)\implies(21),(14)\implies(22),(15)\implies(23),(16)\implies(24),(17)\implies(25)$ and $(18)\implies(26)$. Hence V_{11} of E_2 in Ω. Proved theorem.

Remark 1.

(a) *The natural death rate c_1 of Pathogenic fungus is greater than the monod value,*

(b) *The addition of natural death rate d_2 of Infected Orange Trees, to the product of per capita reproduction rate e_1 of Infected Orange Trees and the population of Herbivore (H) is greater than the twice product of Infected Orange Trees population and it's crowding effect β_2, then only E_2 will exist in the presence of pathogenic fungus.*

7 Global Stability of the equilibrium point E_2 for the Model

Theorem 2: *In the region Ω, if the following conditions hold:*

$$\frac{k_2\tilde{T}_1}{k_1\tilde{T}_2}<\frac{(d_1+a_1P+a_2H)(d_2+e_1H)}{(a_1P)^2} \tag{27}$$
$$\frac{d_2k_1k_3\tilde{T}_2}{a_1k_2}<(d_1+a_1P+a_2H)(C_1-m) \tag{28}$$
$$\frac{a_1d_3^2k_2k_4^2\tilde{T}_1}{d_2k_3\tilde{T}_2}<\beta_3(C_1-m) \tag{29}$$

where,

$$R_1 = \frac{k_2\tilde{T}_1}{k_1\tilde{T}_2} \tag{30}$$

$$R_2 = \frac{\tilde{T}_1}{k_1} \tag{31}$$

$$R_3 = \frac{a_1k_2\tilde{T}_1^{\,2}}{d_2k_1k_3\tilde{T}_2} \tag{32}$$

then E_2 will be globally asymptotically stable in the region Ω.

Proof: Let us consider a positive definite function V_1 for $E_2(T_1, T_2, H, P)$ as:

$$V_1 = (T_1 - \tilde{T}_1)^2 + R_1(T_2 - \tilde{T}_2)^2 + R_2\left(H - \tilde{H} - \tilde{H}log\left(\frac{H}{\tilde{H}}\right)\right) + R_3(P - \tilde{P})^2$$

Differentiating V_1 w.r.t. t, we obtain

$$\frac{dV_1}{dt} = (T_1 - \tilde{T}_1)\frac{dT_1}{dt} + R_1(T_2 - \tilde{T}_2)\frac{dT_2}{dt} + R_2\left(\frac{H - \tilde{H}}{H}\right)\frac{dH}{dt} + R_3(P - \tilde{P})\frac{dP}{dt}$$

from (1)-(4)

$$\begin{aligned}\frac{dV_1}{dt} = & -d_1(T_1 - \tilde{T}_1)^2 - a_1P(T_1 - \tilde{T}_1)^2 - a_1T_1(T_1 - \tilde{T}_1)(P - \tilde{P}) - a_2H(T_1 - \tilde{T}_1)^2 \\ & -a_2\tilde{T}_1(T_1 - \tilde{T}_1)(H - \tilde{H}) - \beta_1(T_1 - \tilde{T}_1)^2(T_1 + \tilde{T}_1) + R_1a_1P(T_1 - \tilde{T}_1)(T_2 - \tilde{T}_2) \\ & +R_1a_1\tilde{T}_1(P - \tilde{P})(T_2 - \tilde{T}_2) - R_1d_2(T_2 - \tilde{T}_2)^2 - R_1e_1H(T_2 - \tilde{T}_2)^2 \\ & -R_1e_1\tilde{T}_2(T_2 - \tilde{T}_2)(H - \tilde{H}) - R_1\beta_2(T_2 - \tilde{T}_2)^2(T_2 + \tilde{T}_2) \\ & +R_2k_1a_2(T_1 - \tilde{T}_1)(H - \tilde{H}) + R_2k_2e_1(T_2 - \tilde{T}_2)(H - \tilde{H}) - R_2\beta_3(H - \tilde{H})^2 \\ & +R_3m(P - \tilde{P})^2 + R_3k_3d_2(T_2 - \tilde{T}_2)(P - \tilde{P}) - c_1R_3(P - \tilde{P})^2 \\ & +R_3k_4d_3(H - \tilde{H})(P - \tilde{P})\end{aligned}$$

Now $\frac{dV_1}{dt}$ can be written in the quadratic forms:

$$\begin{aligned}\frac{dV_1}{dt} \leq & \ (d_1 + a_1P + a_2H)(T_1 - \tilde{T}_1)^2 - (d_2R_1 + R_1e_1H)(T_2 - \tilde{T}_2)^2 \\ & -R_2\beta_3(H - \overline{H})^2 - (c_1R_3 - R_3m)(P - \overline{P})^2 - a_1\tilde{T}_1(T_1 - \tilde{T}_1)(P - \overline{P}) \\ & +R_1a_1P(T_1 - \tilde{T}_1)(T_2 - \tilde{T}_2) + R_3k_4d_3(H - \overline{H})(P - \overline{P})\end{aligned}$$

where,

$$b_{11} = (d_1 + a_1P + a_2H), \qquad b_{22} = (d_2R_1 + R_1e_1H)$$

$$b_{33} = R_2\beta_3 \qquad b_{44} = (c_1R_3 - R_3m)$$
$$b_{12} = (R_1a_1P) \qquad b_{14} = (a_1\tilde{T}_1)$$
$$b_{34} = (R_3k_4d_3) \qquad R_1 = \frac{k_2\tilde{T}_1}{k_1\tilde{T}_2}$$
$$R_2 = \frac{\tilde{T}_1}{k_1} \qquad R_3 = \frac{a_1k_2\tilde{T}_1}{d_2k_1k_3\tilde{T}_2}$$

By the Sylvester's rule, we obtain that $\frac{dV_1}{dt}$ will be negative function with the inequalities:

$$b_{11} > 0 \tag{33}$$
$$b_{22} > 0 \tag{34}$$
$$b_{33} > 0 \tag{35}$$
$$b_{44} > 0 \tag{36}$$
$$b_{11}b_{22} > b_{12}^2 \tag{37}$$
$$b_{11}b_{44} > b_{14}^2 \tag{38}$$
$$b_{33}b_{44} > b_{34}^2 \tag{39}$$

We note that the inequalities, $(25) \implies (35), (26) \implies (36)$ and $(27) \implies (37)$. Hence V_1 is Liapunov function for E_2 in Ω, proving the theorem.

8 Numerical Simulation

For numerical results, we have used MATLAB software. The numerical simulations support the analytical findings. The figures carry the locally asymptotically stability of all the equilibriums of the given Mathematical Model.

(a) **Graph for equilibrium point E_1 is obtained considering following parametric values:**
$\Lambda = 0.9; d_1 = 0.29; d_2 = 0.3; d_3 = 2.2; a_1 = 0.9; a_2 = 0.9; e_1 = 0.8; k_1 = 0.5;$
$k_2 = 0.9; c_1 = 0.4; k_3 = 0.9; \beta_1 = 0.05; \beta_2 = 0.5; \beta_3 = 0.05; k_4 = 0.90; m = 0.008;$

(b) **Graph for equilibrium point E_2 is obtained considering following parametric values:**
$\Lambda = 0.8; d_1 = 0.06; c_1 = 0.8; k_4 = 0.5; a_1 = 0.3; a_2 = 0.3; e_1 = 0.3; k_1 = 1.5;$
$k_2 = 0.7; k_3 = 0.5; \beta_1 = 0.0005; \beta_2 = 0.005; \beta_3 = 0.005; m = 0.0005; d_2 = 0.01;$
$d_3 = 0.65;$

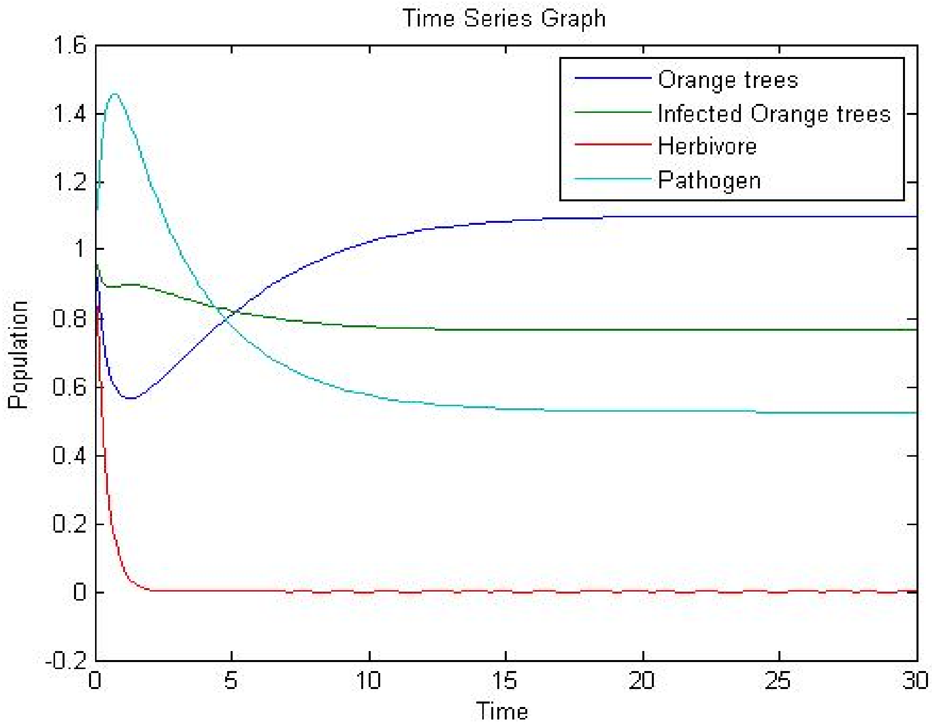

Figure 2: Graph for E_1 in the presence of Orange tree, Infected Orange tree, Pathogen and absence of Herbivore.

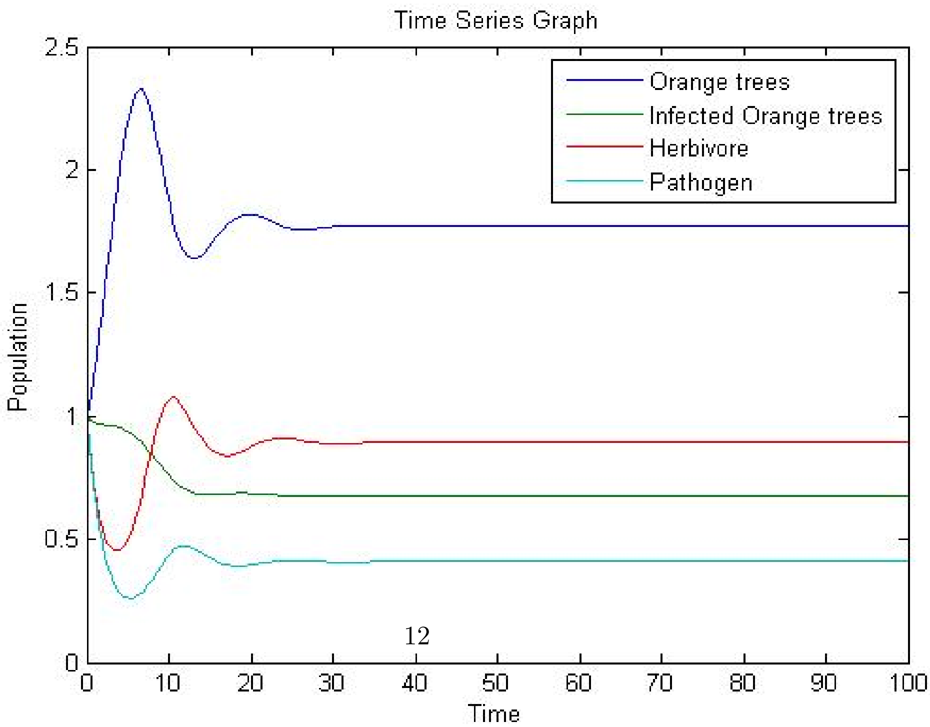

Figure 3: Graph for E_2 in the presence of Orange trees, Infected Orange trees, Herbivore and Pathogen.

(c) Table for equilibrium point E_1

E_1 $(T_1, T_2, 0, P)$ Equilibrium point.

Points	T_1	T_2	H	P
Case 1	1.0997	0.7636	0	0.5260
Case 2	2.2391	0	0	0

Table 2: Numerical equilibrium values for different cases of E_1. In Case 1, only H is absent. In Case 2, T_2, H, P are absent.

(d) Table for equilibrium point E_2

E_2 (T_1, T_2, H, P) Equilibrium point.

Points	T_1	T_2	H	P
Case 1	1.7804	0.6979	0.9175	0.3775
Case 2	1.4626	0	1.6209	0
Case 3	12.1021	0	0	0

Table 3: Numerical equilibrium values for different cases of E_2. In Case 1, all the variables are present. In Case 2, only T_2 and P are absent. In Case 3, only T_1 is present.

9 Numerical Simulation

One of the major threats to secure food supplies and sustainable agricultural output is the increase of aggressive pathogens [1, 2, 3]. In this paper, we have studied a mathematical model to investigate the biological manage of orange trees in the presence of pathogens and Herbivores. The model has at most two equilibrium points, $E_1(\overline{T_1}, \overline{T_2}, 0, \overline{P})$ and $E_2(\tilde{T_1}, \tilde{T_2}, \tilde{H}, \tilde{P})$. It has been observed from the stability of E_1 that Orange trees will survive, even in the presence of pathogens when herbivores are not present in the system.

It has been observed from the stability of E_2 that the Orange trees will survive, even in the presence of herbivores and pathogens.

From the Remarks 1 and 2, and stability conditions of E_1 and E_2 it has been observed that the role of Monod function is more appropriate in the study of orange trees dynamics [see parameter values]. The interior equilibrium point E_2 of mathematical model is locally and also globally stable showing the co-existence. However, from the equilibrium values (see Table 2 for E_1) it is seen that the equilibrium density of Orange trees increases in the absence of pathogen and herbivore. Also from the equilibrium values (see Table 3 for E_2), it is seen that the equilibrium density of healthy Orange trees reduces due to the presence of pathogenic fungus and herbivores. In the pathogenic fungus and also in the absence of herbivores, the density of healthy orange trees is high; where as, if herbivore population is present then the density of healthy orange trees reduces due to the preying activity even if pathogenic fungus is absent.

Further, it is also noted that the herbivore population reduces when pathogenic fungus is present due to infected orange trees on which herbivores are preying.

Declaration of conflict interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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