

Mathematical implications of the effect of toxicants and distributed delay on tri-trophic food chain model

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Abstract This work creates and examines a mathematical model that is nonlinear to investigate the effects of toxins on a marine biological food chain system that comprises three species. The distributed delay is supposed to be taken into account in the environment toxicant in the model. Even though its structure is simple, the delay differential equation has a large range of outcomes. The model consists of five state variables namely phytoplankton, zooplankton, fish, environmental toxin, and distributed delay. The model is being analyzed for stability by using the Jacobian matrix and Liapunov function. From the analysis, we obtained sufficient constraints for local and global stability. It has been observed through the analysis that the effect of toxics on the ecosystem has a significant impact. Additionally, a study of the Hopf bifurcation has been conducted concerning a number of important characteristics. Finally, our analytical findings are confirmed by numerical simulations.

Keywords: Tri-trophic food chain, Toxin, Routh- Hurwitz's, Stability discussion, Hopf bifurcation, Distributed delay.

Mathematics Subject Classification 92D25, 92D40, 93D20, 34D23, 34C23

1 Introduction

Marine ecosystems have been adversely affected by industrial discharges, run-off from agriculture and urban areas, accidental spills, etc. In December 2023, R. Vani Ganapathy et. al. [3] showed that Tamil Nadu experienced a significant environmental disaster when an oil spill occurred at Ennore Creek near Chennai. This environmental disaster disrupted the livelihoods of fishermen, posed health risks to residents, and caused extensive ecological damage, including harm to aquatic life and sensitive mangrove ecosystems. Many species are exposed to various toxicants, which can affect their growth rates, carrying capacities, and available resources. Phytoplankton offer significant benefits to Earth. They serve as a primary food source for marine life, produce oxygen for humans, and absorb half of the carbon dioxide in the atmosphere. Zooplankton are small animals within the planktonic community that consume phytoplankton and serve as a highly favored food source for fish. Toxicants could play a crucial role in reshaping the abundance of species and the structure of ecological communities.

Ecologists and mathematicians frequently use food chain mathematical models to illustrate the eating relationships among prey-predator populations within their ecosystems or environment. There is a significant interest in the study of prey-predator models, particularly in ecosystems involving tri-trophic species [4, 6, 14]. Some prior research has explored tri-trophic mathematical food chain models, focusing on the consequences of toxicants on the survival of food chain species through the use of mathematical equations. Misra and Raveendra [13] have discussed the system-level impacts of toxicants while looking at a system of tri-trophic species food chain model considered Holling type II functional responses. Hallam et al. [7] discussed and investigated the effects of toxicants on populations and determined that the presence of toxicants might lead to the emergence of multiple stable equilibrium positions. In contrast, when there is no effect of toxicants, population survival displays only a single stable equilibrium. Panja and Mondal [15] have investigated the stability and coexistence of phytoplankton population, zooplankton population, and fish population. Arindam Mandal et al. [11] studied a mathematical model to raise public awareness in order to limit the harmful effects of toxicants on the phytoplankton-zooplankton system. Their research showed that the phytoplankton-zooplankton system may become unstable when there is a moderate concentration of anthropogenic contaminants present. However, public awareness induces stability in the contaminated system. R.K. Upadhyay et al. [19] investigate the dynamic complexity of two different chaotic behavior in tri-trophic aquatic prey-predator model systems that replicate real-world maritime conditions. From this study, they concluded that the diverse mortality rate functions of zooplankton resulting from toxin-producing phytoplankton play a crucial role in regulating oscillations, determining coexistence, and influencing the survival rate or extinction rate of the zooplankton species. An oil or liquid spill, mainly in the marine ecological environment can significantly threaten marine life. Huda et al. [5] considered an oil spill prey-predator mathematical interaction model to study the environmental consequences of liquid spills in marine ecosystems. Khadim and Majeed [8] put forward a mathematical model to examine the impact of defensive and auxiliary mechanisms involving the secretion of toxic substances by organisms on the food chain system of three species. This analysis takes into account the presence of harvest efforts in the ecosystem. In a marine food chain experimental study conducted by Zhang et al. [20] the fluctuation of accumulated mercury (Hg) in microalgae species, along with the selection of zooplankton towards algal diets, led to diverse mercury accumulation levels in three Zooplankton species. This highlights the significant influence of feeding selectivity on the transfer of methylmercury (MeHg) along marine food chains. Panja et al. [16] proposed a model incorporating toxicants to study the interactions within a three-species predator-prey system involving phytoplankton, zooplankton, and fish. This model aims to analyze how toxic substances influence the dynamics and relationships among these species in their ecological environment. The investigations mentioned are focused on the immediate impacts of toxicants on species. However, the absorption of environmental toxicants by the resource biomass might not occur instantly. Therefore, incorporating a delay effect in such scenarios is not only logical but also aligns closely with real-world systems.

It is essential to include continuous or distributed time delays in the model since the dynamic behavior of biological systems frequently depends on their past. This approach is essential for studying a wide range of biological processes comprehensively. Over the past few decades, numerous researchers have developed and examined mathematical models incorporating distributed delays in the growth dynamics of biological species. Amit Sharma et al. [17] conducted a study on a mathematical model depicting the interaction between toxin-producing phytoplankton and zooplankton. The model incorporates a delay in zooplankton predation, and the research investigates its impact on the overall dynamics of the phytoplankton-zooplankton interaction. It is shown that time delay can destabilize

the given system and induce oscillation in the population due to Hopf-bifurcation. Das and Ray [2] investigated a phytoplankton-zooplankton system, incorporating nutrients as a delay parameter. According to their research, the stability of the system is unaffected by a delay in the breakdown of dead phytoplankton caused by senescence and other mortalities. Misra and Raveendra [12] put forth a mathematical model that included a delay in the toxicant uptake process by the prey population in order to examine the impact of a toxin in a tri-trophic biological food chain. Chatterjee and Weihua [1] discuss the development and analysis of a mathematical model for the interactions between one phytoplankton species and two zooplankton species. The main goal is to identify the key parameters that influence the system and determine appropriate ranges for these parameters to ensure the stable coexistence of all species in ecosystems with and without delays. In the no-delay system, it has been demonstrated that numerous steady states exist and are stable. In the delay system, the focus is on examining the Hopf bifurcation and determining whether it is supercritical or subcritical, which affects the stability and nature of the oscillations in the system. Ravikant et al. [18] discusses the study of a nutrient-plankton system that incorporates the effects of time delays in nutrient recycling and toxin-dependent responses. This study extends to a model involving phytoplankton, zooplankton, and fish populations, considering the impact of environmental toxins. The model includes a distributed delay to account for environmental toxins' effects.

With the view above, we studied a phytoplankton-zooplankton-fish population model with environmental toxicants. Considering the work done by [16], a distributed-time delay has been introduced into environmental toxicant.

2 Model formulation

A tri-trophic marine food chain of predator-prey interaction model has been considered. In the model, we have incorporated a distributed delay mechanism to account for the impact of environmental toxicants. Let the main variables of the mathematical model be P , the density of Phytoplankton; Z , the density of Zooplankton; F , the density of Fish; E_T , Environmental toxin; and R , distributed delay.

With these main variables, we construct a model and operate the system of equations to examine the impact of distributed delay.

$$\begin{aligned}
 \frac{dP}{dT} &= \frac{rP}{1 + \gamma\gamma_1 P E_T} \left(1 - \frac{P}{K}\right) - \frac{\beta P Z}{\alpha + P} \\
 \frac{dZ}{dT} &= \frac{\beta_1 P Z}{\alpha + P} - d_1 Z - \frac{\beta_2 Z F}{jZ^2 + a} \\
 \frac{dF}{dT} &= \frac{\beta_3 Z F}{jZ^2 + a} - d_2 F \\
 \frac{dE_T}{dT} &= A - \gamma P E_T - d_3 E_T - a_1 P R \\
 \frac{dR}{dT} &= \gamma_2 (E_T - R)
 \end{aligned} \tag{1}$$

and the initial constraints are $P(0) \geq 0$, $Z(0) \geq 0$, $F(0) \geq 0$, $E_T \geq 0$, $R(0) > 0$.

It is considered that the growth rate of Phytoplankton in the absence of environmental toxins is of logistic form. Let r be the intrinsic growth rate of the Phytoplankton population, γ is the contact rate between the Phytoplankton population and toxicant, γ_1 is the proportionality constant, K is the

environmental carrying capacity.

The expression $\frac{\beta PZ}{\alpha + P}$ describes the response for grazing the Phytoplankton population by Zooplankton population, α represents about the half-saturation rate for Holling type II response and β is consumption rate of phytoplankton by zooplankton.

As the phytoplankton population is consumed by the zooplankton population, the growth is directly linked to the rate at which phytoplankton is consumed, i.e., the response for zooplankton and $\frac{\beta_1 PZ}{\alpha + P}$, where β_1 is the conversion rate and d_1 is the natural death rate.

The terms $\frac{\beta_i ZF}{jZ^2 + a}$ ($i=2, 3$) describe the response for grazing of Zooplankton population by Fish population where β_2 is the consumption rate of Zooplankton by Fish, j is the inhibitory effect of Zooplankton against fish, a is considered as the half-saturation constraint of Zooplankton for type III response, β_3 is the conversion rate, d_2 is the natural death of fish, E_T is the environmental toxin, A is the constant growth of Environmental toxicant, d_3 is the depletion rate, a_1 is the removal rate of environmental toxin due to uptake by Phytoplankton.

In the model, we have incorporated a distributed delay, $R(T)$, into the term ' $a_1 P E_T$ '. This term describes the uptake of environmental toxicants by the Phytoplankton species (Khare et al. [9]).

$$R(T) = \int_{-\infty}^t \gamma_2 \exp(-\gamma_2(t-s)) E_T(s) ds$$

where $s \in (-\infty, 0]$, γ_2 is the delay parameter.

We want to bring down the parameters in the system (1) with the following transformation:

$$p = \frac{P}{K}, \quad z = \frac{\beta Z}{r\alpha}, \quad f = \frac{\beta_2 F}{jr}, \quad e_T = K E_T, \quad t = rT, \quad n = KR.$$

After re-scaling the model becomes as follows:

$$\frac{dp}{dt} = \frac{p(1-p)}{1 + \gamma\gamma_1 p e_T} - \frac{pz}{1 + e_1 p} \tag{2}$$

$$\frac{dz}{dt} = \frac{\alpha_1 pz}{1 + e_1 p} - d'_1 z - \frac{\alpha_2 z f}{z^2 + e_2} \tag{3}$$

$$\frac{df}{dt} = \frac{\alpha_3 z f}{z^2 + e_2} - d'_2 f \tag{4}$$

$$\frac{de_T}{dt} = e_3 - \alpha_4 p e_T - d'_3 e_T - \alpha_5 n p \tag{5}$$

$$\frac{dn}{dt} = \gamma'_2 (e_T - n) \tag{6}$$

and the initial constraints are $p(0) \geq 0, z(0) \geq 0, f(0) \geq 0, e_T \geq 0, n(0) > 0$.

Here,

$$e_1 = \frac{K}{\alpha}, \quad \alpha_1 = \frac{\beta_1 K}{\alpha r}, \quad d'_1 = \frac{d_1}{r}, \quad e_2 = \frac{a\beta^2}{jr^2\alpha^2}, \quad e_3 = \frac{KA}{r}, \quad \alpha_3 = \frac{\beta\beta_3}{j\alpha r^2}, \quad \alpha_4 = \frac{\gamma K}{r}, \quad d'_3 = \frac{d_3}{r},$$

$$\alpha_5 = \frac{a_1 K}{r}, \quad \gamma'_2 = \frac{\gamma_2}{r}$$

All these parameters, of course, assume only positive values.

Lemma 1. Suppose n is a positive integer and $f_i(t, x_1, x_2, \dots, x_n)$, ($i=1, 2, 3, \dots, n$) are smooth functions. If $f_i|_{x_i=0, \Omega \in R^n_+} \geq 0$ (where $\Omega = (x_1, x_2, \dots, x_n)^T \in R^n$), then R^n_+ is an invariant domain of the following equations:

$$\frac{dx_i}{dt} = f_i(t, x_1, x_2, \dots, x_n), (i = 1, 2, 3, \dots, n)$$

If $f_i|_{x_i=0, \Omega \in R^n} \leq 0$ (where $\Omega = (x_1, x_2, \dots, x_n)^T \in R^n$), then R^n_{-0} is an invariant domain of the above equations [21].

Theorem 1.

$$R^5_{+0} = \{(p, z, f, e_T, n)^T | p \geq 0, z \geq 0, f \geq 0, e_T \geq 0, n \geq 0\}$$

is an invariant domain of Eqs. (2)-(6).

Proof: Denote that $\Omega := (p, z, f, e_T, n)^T$. For the model, we observed that

$$g_1|_{p=0, \Omega \in R^5_{+0}} = \frac{p(1-p)}{1 + \gamma\gamma_1 p e_T} - \frac{pz}{1 + e_1 p} = 0 \geq 0 \tag{7}$$

$$g_2|_{z=0, \Omega \in R^5_{+0}} = \frac{\alpha_1 pz}{1 + e_1 p} - d'_1 z - \frac{\alpha_2 z f}{z^2 + e_2} = 0 \geq 0 \tag{8}$$

$$g_3|_{f=0, \Omega \in R^5_{+0}} = \frac{\alpha_3 z f}{z^2 + e_2} - d'_2 f = 0 \geq 0 \tag{9}$$

$$g_4|_{e_T=0, \Omega \in R^5_{+0}} = e_3 - \alpha_4 p e_T - d'_3 e_T - \alpha_5 n p = e_3 \geq 0 \tag{10}$$

$$g_5|_{n=0, \Omega \in R^5_{+0}} = \gamma'_2 (e_T - n) = \gamma'_2 e_T \geq 0 \tag{11}$$

By the above lemma, we can conclude that R^5_{+0} is an invariant domain of Eqs. (2)-(6).

3 Boundedness of the Model

Here we prove the boundedness of the model solutions, thereby confirming the ecologically meaningful.

Theorem 2 The set

$$\Omega = \{(p, z, f, e_T, n) \in R^5 : 0 \leq p \leq 1, 0 \leq z \leq \frac{1}{\alpha_1} + \frac{\alpha_2}{\alpha_3 \alpha_1} f \leq K_1, 0 \leq e_T \leq K_2, 0 \leq e_T + n \leq K_3\}$$

where $K_1 = \frac{1}{\phi_1}$, $\phi_1 = \min\{1, d'_1, d'_2\}$, $K_2 = \frac{e_3}{d'_3}$, $K_3 = \frac{e_3}{\phi_2}$, $\phi_2 = \min\{d'_3 - \gamma'_2, \gamma'_2\}$, is a region of desirability for all the solutions institute in the interior of the non-negative region.

Proof: From equation (2) we get,

$$\frac{dp}{dt} = \frac{p(1-p)}{1 + \gamma\gamma_1 p e_T} - \frac{pz}{1 + e_1 p}$$

$$\frac{dp}{dt} \leq \frac{p(1-p)}{1 + \gamma\gamma_1 p e_T} \leq p(1-p)$$

$$\frac{dp}{p(1-p)} \leq dt$$

As $t \rightarrow \infty$, we get, $p \leq 1$.

Now, let us consider a function

$$W = p + \frac{1}{\alpha_1} z + \frac{\alpha_2}{\alpha_3 \alpha_1} f$$

by using Eqs. (2)-(6), we get

$$\frac{dW}{dt} + \phi_1 W \leq 1$$

where $\phi_1 = \min\{1, d'_1, d'_2\}$, now we get $W \leq \frac{1}{\phi_1}$ as $t \rightarrow \infty$
From eq.(5)

$$\frac{de_T}{dt} \leq e_3 - d'_3 e_T$$

$e_T \leq \frac{e_3}{d'_3}$ as $t \rightarrow \infty$.

Now Let

$$\begin{aligned} W_1 &= e_T + n \\ \frac{dW_1}{dt} &= \frac{de_T}{dt} + \frac{dn}{dt} \\ \frac{dW_1}{dt} + \phi_2 W_1 &\leq e_3 \end{aligned}$$

where $\phi_2 = \min\{d'_3 - \gamma'_2, \gamma'_2\}$, then we get

$$W_1 \leq \frac{e_3}{\phi_2}$$

Hence all the solutions of the model are bounded in Ω .

4 Analysis of the Model

4.1 Equilibrium points

The Model has the following three positive equilibrium points namely, $\hat{E}_1(\hat{p}, 0, 0, \hat{e}_T, \hat{n})$, $\bar{E}_2(\bar{p}, \bar{z}, 0, \bar{e}_T, \bar{n})$, $E_3^*(p^*, z^*, f^*, e_T^*, n^*)$. We prove the existence of $\hat{E}_1, \bar{E}_2, E_3^*$ as follows:

Existence of $\hat{E}_1 = (\hat{p}, 0, 0, \hat{e}_T, \hat{n})$ from (6),

$$\hat{e}_T = \hat{n}$$

from (2) and (5),

$$\hat{p} = 1, \quad \hat{e}_T = \frac{e_3}{\alpha_4 + d'_3 + \alpha_5}$$

then the equilibrium point is $\hat{E}_1 \left(1, 0, 0, \frac{e_3}{\alpha_4 + d'_3 + \alpha_5}, \frac{e_3}{\alpha_4 + \alpha_5 + d'_3} \right)$.

Existence of $\bar{E}_2 = (\bar{p}, \bar{z}, 0, \bar{e}_T, \bar{n})$ from (6),

$$\bar{e}_T = \bar{n}$$

from (2),

$$\bar{p} = \frac{d'_1}{\alpha_1 - d'_1 e_1}, \quad \bar{p} > 0 \text{ if } \alpha_1 > d'_1 e_1 \tag{12}$$

from (5),

$$\bar{e}_T = \frac{e_3}{(\alpha_4 + \alpha_5)\bar{p} + d'_3} \tag{13}$$

from (2),

$$\bar{z} = \frac{(1 - \bar{p})(1 + e_1\bar{p})}{1 + \gamma\gamma_1\bar{p}e_T}, \quad \text{if } \bar{p} < 1 \tag{14}$$

where the value of \bar{p} and \bar{e}_T can be calculated by using eq. (12) and (13).

Existence of $E_3^* = (p^*, z^*, f^*, e_T^*, n^*)$

from (6),

$$e_T^* = n^*$$

from (4),

$$z^* = \frac{\alpha_3 \pm \sqrt{\alpha_3^2 - 4d_2'^2 a_1}}{2d_2'} \tag{15}$$

$z^* > 0$ if $\alpha_3^2 - 4d_2'^2 a_1 > 0$.

from (2),

$$f^* = \frac{\alpha_3 z^*}{d_2' \alpha_2} \left(\frac{\alpha_1 p^*}{1 + e_1 p^*} - d_1' \right) \tag{16}$$

provided $\frac{\alpha_1 p^*}{1 + e_1 p^*} > d_1'$.

from (5),

$$e_T^* = \frac{e_3}{(\alpha_4 + \alpha_5)p^* + d_3'} \tag{17}$$

by using the value of z^* and e_T^* in eq.(2), we get a cubic equation in p^* ,

$$e_1(\alpha_4 + \alpha_5)p^{*3} + (e_1 d_3' + (1 - e_1)(\alpha_4 + \alpha_5))p^{*2} + ((\alpha_4 + \alpha_5 + \gamma\gamma_1 e_3)z^* + (1 - e_1)d_3' - (\alpha_4 + \alpha_5))p^* + (z^* d_3' - d_3') = 0$$

by solving the above equation for p^* we get the positive solution.

4.2 Local stability of the Model

For the study of local stability, the equilibrium points in the model are assessed by calculating the eigenvalues with the variational matrix around these equilibrium points.

The general variational matrix for the model

$$J = \begin{bmatrix} c_{11} & \frac{-p}{(1+e_1p)} & 0 & c_{14} & 0 \\ \frac{\alpha_1 z}{(1+e_1p)^2} & c_{22} & \frac{-\alpha_2 z}{z^2+a_1} & 0 & 0 \\ 0 & c_{32} & c_{33} & 0 & 0 \\ -\alpha_4 e_T - \alpha_5 n & 0 & 0 & -\alpha_4 p - d_3' & -\alpha_5 p \\ 0 & 0 & 0 & \gamma_2' & -\gamma_2' \end{bmatrix}$$

where,

$$c_{11} = \frac{1 - 2p - \gamma\gamma_1 e_T p^2}{(1 + \gamma\gamma_1 e_T p)^2} - \frac{z}{(1 + e_1 p^2)}, \quad c_{14} = \frac{-\gamma\gamma_1 p^2 (1 - p)}{(1 + \gamma\gamma_1 p e_T)^2}, \quad c_{33} = \frac{\alpha_3 z}{(z^2 + a_1)} - d_2'$$

$$c_{22} = \frac{\alpha_1 p}{(1 + e_1 p)} - d_1' - \frac{(a_1 - z^2)\alpha_2 f}{(z^2 + a_1)^2}, \quad c_{32} = \frac{(a_1 - z^2)\alpha_3 f}{(z^2 + a_1)^2}$$

Local stability of $\hat{E}_1(\hat{p}, 0, 0, \hat{e}_T, \hat{n})$:

For this equilibrium point, the $|J - \lambda I|$ would be

$$|J - \lambda I| = \left(\frac{1 - 2\hat{p} - \gamma\gamma_1 e_T' \hat{p}^2}{(1 + \gamma\gamma_1 e_T' \hat{p})^2} - \lambda \right) \left(\frac{\alpha_1 \hat{p}}{(1 + e_1 \hat{p})} - d_1' - \lambda \right) (-d_2' - \lambda) \times [(\gamma_2' + \lambda)(\alpha_4 \hat{p} + d_3' + \lambda) + \alpha_5 \hat{p} \gamma_2']$$

then the eigenvalues of the characteristic equation are

$$\frac{-1}{(1 + \gamma\gamma_1 e_T')}, \frac{\alpha_1}{1 + e_1} - d_1', -d_2', \lambda^2 + \lambda(\gamma_2' + \alpha_4 + d_3') + (\alpha_5 \gamma_2' + \gamma_2' \alpha_4 + \gamma_2' d_3') = 0.$$

Hence, the point \hat{E}_1 is locally asymptotically stable if

$$\frac{\alpha_1}{1 + e_1} < d_1' \implies \frac{\alpha}{d_1'} < (1 + e_1) \implies \left(\frac{\beta_1}{d_1} - \frac{\alpha}{k} \right) < 1 \implies \frac{\beta_1}{d_1} < 1$$

and

$$\lambda = \frac{-A_1 \pm \sqrt{A_1^2 - 4A_2}}{2} \quad \text{where} \quad A_1 = \gamma_2' + \alpha_4 + d_3', \quad A_2 = \alpha_5 \gamma_2' + \gamma_2' \alpha_4 + \gamma_2' d_3'$$

$$A_2 > 0, \quad A_1^2 \geq 4A_2.$$

Remark 1. From the stability conditions of \hat{E}_1 , it may be noted that the conversion rate of phytoplankton into zooplankton is less than the death rate of zooplankton.

Local stability of $\bar{E}_2(\bar{p}, \bar{z}, 0, \bar{e}_T, \bar{n})$:

For this equilibrium point, the $|J - \lambda I|$ would be

$$\lambda^4 + \psi_1 \lambda^3 + \psi_2 \lambda^2 + \psi_3 \lambda + \psi_4 = 0$$

where the parametric values of $\psi_1, \psi_2, \psi_3, \psi_4$ is given as

$$\begin{aligned} \psi_1 &= \gamma_2' + \alpha_4 \bar{p} + d_3' + d_1' + \phi_2' - \alpha_1 \phi_3 - \phi_1 \\ \psi_2 &= [(\phi_1 - \phi_2)(\alpha_1 \phi_3 - d_1') + \phi_3 \alpha_1 \phi_2] + [(\alpha_4 \bar{p} + d_3') \gamma_2' + \alpha_5 \gamma_2' \bar{p}] - \phi_5 (\bar{e}_T \alpha_4 + \alpha_5 \bar{n}) \\ &\quad - (\alpha_1 \phi_3 - d_1' + \phi_1 - \phi_2)(\gamma_2' + \alpha_4 \bar{p} + d_3') \\ \psi_3 &= ((\phi_1 - \phi_2)(\alpha_1 \phi_3 - d_1') + \phi_3 \alpha_1 \phi_2)(\gamma_2' + \alpha_4 \bar{p} + d_3') - (\alpha_1 \phi_3 - d_1' + \phi_1 - \phi_2)[(\alpha_4 \bar{p} \\ &\quad + d_3') \gamma_2' + \alpha_5 \gamma_2' \bar{p}] - \phi_5 (\alpha_4 \bar{e}_T + \alpha_5 \bar{n}) \gamma_2' + (\bar{e}_T \alpha_4 + \alpha_5 \bar{n})(\phi_3 \alpha_2 \phi_4 + \phi_5 (\alpha_1 \phi_3 - d_1')) \\ \psi_4 &= ((\phi_1 - \phi_2)(\alpha_1 \phi_3 - d_1') + \phi_3 \alpha_1 \phi_2)[(\alpha_4 \bar{p} + d_3') \gamma_2' + \alpha_5 \gamma_2' \bar{p}] + (\bar{e}_T \alpha_4 + \alpha_5 \bar{n}) \gamma_2' (\phi_3 \alpha_2 \\ &\quad \phi_4 + \phi_5 (\alpha_1 \phi_3 - d_1')) \end{aligned}$$

where

$$\phi_1 = \frac{1 - 2\bar{p} - \gamma\gamma_1 \bar{e}_T \bar{p}^2}{(1 + \gamma\gamma_1 \bar{e}_T \bar{p})^2}, \phi_2 = \frac{\bar{z}}{(1 + e_1 \bar{p})^2}, \phi_3 = \frac{\bar{p}}{1 + e_1 \bar{p}}, \phi_4 = \frac{\bar{z}}{\bar{z} + e_2}, \phi_5 = \frac{\gamma\gamma_1 \bar{p}^2 (1 - \bar{p})}{(1 + \gamma\gamma_1 \bar{e}_T \bar{p})^2}.$$

By Routh-Hurwitz criterion, the point \bar{E}_2 is locally asymptotically stable if $\psi_i > 0$ ($i=1,2,3,4$), $\bar{p} \leq 1, \alpha_1 \phi_3 - d_1' > 0 \implies 1 + \frac{\alpha}{k\bar{p}} < \frac{\beta_1}{d_1}, \implies \frac{\beta_1}{d_1} > 1,$

$\psi_1 \psi_2 - \psi_3 > 0$ and $\psi_1 \psi_2 \psi_3 - \psi_1^2 \psi_4 - \psi_3^2 > 0$ holds.

Remark 2. From the stability condition of \bar{E}_2 , the following observation can be made:

- (i) The equilibria of phytoplankton should be less than one.
- (ii) The ratio of the conversion rate of phytoplankton to zooplankton to the natural death rate of zooplankton is greater than one then only phytoplankton and zooplankton will survive when the distributed delay is considered.

Local stability of $E_3^*(p^*, z^*, f^*, e_T^*, n^*)$:

The characteristic equation is given by

$$\lambda^5 + R_1 \lambda^4 + R_2 \lambda^3 + R_3 \lambda^2 + R_4 \lambda + R_5 = 0 \tag{18}$$

where the parametric values of R_1, R_2, R_3, R_4, R_5 is given as

$$\begin{aligned} R_1 &= \gamma_2' + \alpha_4 p^* + d_3' - d_2' - d_1' - w_2 + \alpha_3 w_6 + w_1 - \alpha_2 w_5 + \alpha_1 w_3 \\ R_2 &= (\alpha_4 p^* + d_3') \gamma_2' + \alpha_5 p^* \gamma_2' + (\alpha_4 p^* + d_3' + \gamma_2')(w_1 - w_2 + \alpha_1 w_3 - d_1' - d_2' - \alpha_2 w_5 \\ &\quad + \alpha_3 w_6) + (w_1 - w_2)(\alpha_3 w_6 - d_1' - d_2' - \alpha_2 w_5 + w_3 w_2 \alpha_1)(\alpha_3 w_6 - d_2')(\alpha_1 w_3 - d_1' \\ &\quad - \alpha_2 w_5) \\ R_3 &= -(w_1 - w_2 + \alpha_1 w_3 - d_1' - d_2' - \alpha_2 w_5 + \alpha_3 w_6)((\alpha_4 p^* + d_3') \gamma_2' + \alpha_5 p^* \gamma_2') + (\alpha_4 p^* \\ &\quad + d_3' + \gamma_2')((\alpha_3 w_6 - d_2')(\alpha_1 w_3 - d_1' - \alpha_2 w_5) + (w_1 - w_2)((\alpha_3 w_6 - d_1' - d_2' - \alpha_2 w_5 \\ &\quad + w_3 w_2 \alpha_1)) - w_4(\alpha_4 + \alpha_5) e_T^* \\ R_4 &= -(\alpha_4 p^* + d_3' \gamma_2')((w_1 - w_2)(\alpha_3 w_6 - d_2')(-\alpha_2 w_5 + \alpha_1 w_3 - d_1') + w_6 w_5 \alpha_2 \alpha_3 + w_3 w_2 \\ &\quad \alpha_1(w_6 \alpha_3 - d_2')) + ((\alpha_4 p^* + d_3') \gamma_2' + \alpha_5 \gamma_2' p^*)((w_1 - w_2)(\alpha_1 w_3 - d_1' - d_2' - \alpha_2 w_5 + \\ &\quad w_6 \alpha_3 + \alpha_1 w_3 w_2)) + (\alpha_3 w_6 - d_2')(-\alpha_2 w_5 + \alpha_1 w_3 - d_1') + w_4(\alpha_4 + \alpha_5) e_T^*((w_3 \alpha_1 - \\ &\quad d_1' - w_5 \alpha_2) \gamma_2' + w_6 w_5 \alpha_2 \alpha_3) \\ R_5 &= w_4(\alpha_4 + \alpha_5) e_T^*((w_3 \alpha_1 - d_1' - w_5 \alpha_2) \gamma_2' + w_6 w_5 \alpha_2 \alpha_3) - ((d_3' + \alpha_4 p^*) \gamma_2' + \alpha_5 p^* \gamma_2') \\ &\quad ((w_1 - w_2)(\alpha_3 w_6 - d_2')(\alpha_1 w_3 - d_1' - w_5 \alpha_2) + w_6 w_5 \alpha_2 \alpha_3 + w_3 w_2 \alpha_1(w_6 \alpha_3 - d_2')) \end{aligned}$$

where

$$\begin{aligned} w_1 &= \frac{1-2p^*-\gamma_1 e_T^* p^{*2}}{(1+\gamma_1 e_T^* p^*)^2}, & w_2 &= \frac{z^*}{(1+e_1 p^*)^2}, & w_3 &= \frac{p^*}{1+e_1 p^*}, & w_4 &= \frac{\gamma_1 p^{*2}(1-p^*)}{(1+\gamma_1 e_T^* p^*)^2}, \\ w_5 &= \frac{(e_2-z^{*2})f^*}{(z^{*2}+e_2)^2}, & w_6 &= \frac{z^*}{(z^{*2}+e_2)^2} \end{aligned}$$

According to the Routh-Hurwitz criterion, $E_3^*(p^*, z^*, f^*, e_T^*, n^*)$ is locally asymptotically stable if $p^* < 1, w_6 \alpha_3 - d_2' > 0 \implies \beta \beta_3 z^* > d_2(z^{*2} + e_2)^2 j \alpha,$

$e_2 > z^{*2} \implies z^{*2} < \frac{a \beta^2}{j r^2 \alpha^2}$ and $R_i > 0, i = 1, 2, 3, 4, 5, R_1 R_2 R_3 > R_3^2 + R_1^2 R_4$ and

$(R_1 R_4 - R_5)(R_1 R_2 R_3 - R_3^2 - R_1^2 R_4) > R_5(R_1 R_2 - R_3)^2 + R_1 R_5^2.$

Remark 3. The following observation can be made from the stability of E_3 :

- (i) The equilibria of phytoplankton is less than one.
- (ii) The square of equilibria of zooplankton is less than the ratio of the square of consumption rate of phytoplankton by zooplankton, half saturation constraints of zooplankton, to the square of the intrinsic growth rate of phytoplankton, the inhibitory effect of zooplankton against fish and the square of half-saturation rate of phytoplankton.

Interpreting the outputs in the ecological environment based on these findings is challenging. Nevertheless, we have verified all these conditions numerically by assessing a range of parametric values.

We determine the circumstances in which the system performs Hopf-bifurcation. To study this, we select γ as a bifurcation parameter. Let's use Liu's rule [10], to ascertain the prerequisites for the formation of periodic solutions with modest amplitudes that result from a Hopf bifurcation.

As γ would be the coefficient of the characteristic equation (18) is a function of γ and we use $R_i = R_i(\gamma)$ for $i = 1, 2, 3, 4, 5$. Pointing the values R_i 's are smooth basis of γ .

If the critical number $\bar{\gamma}$ of γ noted (i) a simple pair of complex conjugate eigenvalues of characteristic equation exists, say, $\lambda_1(\gamma) = x(\gamma) + iy(\gamma)$, $\lambda_2(\gamma) = x(\gamma) - iy(\gamma) = \bar{\lambda}_1(\gamma)$. These eigenvalues become purely imaginary at $\gamma = \bar{\gamma}$, i.e., $\lambda_1(\bar{\gamma}) = iy_0$, $\lambda_2(\bar{\gamma}) = -iy_0$, with $y(\bar{\gamma}) = y_0 > 0$, and rest of the eigenvalues will be real and negative; and (ii) the transversality condition,

$$dRe\lambda_i(\bar{\gamma})/d\gamma|_{\gamma=\bar{\gamma}} = dx(\gamma)/d\gamma|_{\gamma=\bar{\gamma}} \neq 0$$

is satisfied. Then we find at $\gamma = \bar{\gamma}$, a Hopf-bifurcation. Now if we don't have eigenvalues, [19] explained that (observing the output to the present case): if $R_i(\gamma)$,

$$\begin{aligned} \Delta_1(\gamma) &= R_1(\gamma)R_2(\gamma) - R_3(\gamma), \\ \Delta_2(\gamma) &= R_1(\gamma)R_2(\gamma)R_3(\gamma) - (R_3^2(\gamma) + R_1^2(\gamma)R_4(\gamma)), \\ \Delta_3(\gamma) &= [R_3(\gamma)R_4(\gamma) - R_2(\gamma)R_5(\gamma)][R_1(\gamma)R_2(\gamma) - R_3(\gamma)] - [R_1(\gamma)R_4(\gamma) - R_5(\gamma)]^2 \end{aligned}$$

are functions of ' γ ' will be in an open interval containing $\bar{\gamma} \in R^+$ so these condition hold:

$$(i_*) R_1(\bar{\gamma}) > 0, \Delta_1(\bar{\gamma}) > 0, \Delta_2(\bar{\gamma}) > 0, \Delta_3(\bar{\gamma}) = 0;$$

$$(ii_*) d\Delta_3(\gamma)/d\gamma|_{\gamma=\bar{\gamma}} \neq 0$$

then (i_*) and (ii_*) are same to conditions (i) & (ii), a simple Hopf-bifurcation at $\gamma = \bar{\gamma}$. Hence, in a similar condition, we observe the following theorem:

Theorem 3. If a critical value $\bar{\gamma}$ of parameter γ be found such that $R_i(\bar{\gamma}) > 0$, $\Delta_1(\bar{\gamma}) > 0$, $\Delta_2(\bar{\gamma}) > 0$, $\Delta_3(\bar{\gamma}) = 0$ and next $\Delta'_3 \neq 0$ (prime means differentiation w.r.t. γ) then the system (1) undergoes Hopf-bifurcation around E_3^* .

4.3 Global stability

Theorem 4. If the following inequalities hold in the region Ω

$$\sigma_2(1 + \gamma\gamma_1 e_T) > \sigma_1 e_1 z^* \tag{19}$$

$$U_1 \left[d'_1 - \frac{\alpha_1 p}{\sigma_2} (1 - e_1 p^*) \right] > \frac{\alpha_2 f^*}{\sigma_3} (e_2 + z z^*) \tag{20}$$

$$U_2 d'_2 \sigma_3 > \alpha_3 z (z^{*2} + e_2) \tag{21}$$

$$U_3 \left[\frac{1 + \gamma\gamma_1 e_T}{\sigma_1} - \frac{e_1 z^*}{\sigma_2} \right] (d'_3 + \alpha_4 p) > [G_1 + U_3 (\alpha_4 e_T^* + \alpha_5 n^*)]^2 \tag{22}$$

$$U_1 U_2 G_2 [d'_2 \sigma_3 - \alpha_3 z (z^{*2} + e_2)] > \sigma_2 [U_1 \alpha_2 z (z^{*2} + e_2) + G_3]^2 \tag{23}$$

$$U_3 U_4 \gamma'_2 (d'_3 + \alpha_4 p) > [U_3 \alpha_5 p - \gamma'_2 U_4]^2 \tag{24}$$

where,

$$G_1 = \frac{\gamma\gamma_1 p^*}{\sigma_1} (1 - p), \quad \sigma_1 = (1 + \gamma\gamma_1 p e_T)(1 + \gamma\gamma_1 p^* e_T^*), \quad G_3 = U_2 \alpha_3 f^* (z z^* - \alpha_2)$$

$$\sigma_2 = (1 + e_1 p)(1 + e_1 p^*), \quad \sigma_3 = (z^2 + e_2)(z^{*2} + e_2), \quad U_1 = \frac{1 + e_1 p^*}{\alpha_1 z^*} > 0$$

$$G_2 = d'_1 \sigma_2 \sigma_3 - \alpha_1 \sigma_3 p (1 - e_1 p^*) - \sigma_2 \alpha_2 f^* (e_2 + z z^*)$$

then the equilibrium point E_3^* is globally asymptotically stable with respect to all solutions initiating in the interior of the positive region Ω .

Proof. First, let us define a positive definite function for E_3^* :

$$V = \left(p - p^* - p^* \ln \left(\frac{p'}{p^*} \right) \right) + \frac{U_1}{2} (z - z^*)^2 + \frac{U_2}{2} (f - f^*)^2 + \frac{U_3}{2} (e_T - e_T^*)^2 + \frac{U_4}{2} (n - n^*)^2$$

Differentiating the above equation of V with respect to time t , we get

$$\frac{dV}{dt} = \left(\frac{p - p^*}{p} \right) \frac{dp}{dt} + U_1 (z - z^*) \frac{dz}{dt} + U_2 (f - f^*) \frac{df}{dt} + U_3 (e_T - e_T^*) \frac{de_T}{dt} + U_4 (n - n^*) \frac{dn}{dt}$$

Using equations (2)–(6), we get

$$\begin{aligned} \frac{dV}{dt} = & -(p - p^*)^2 \left[\frac{1 + \gamma\gamma_1 e_T}{\sigma_1} - \frac{e_1 z^*}{\sigma_2} \right] - U_2 (f - f^*)^2 \left[d'_2 - \frac{\alpha_3 z}{\sigma_3} ((z^*)^2 + e_2) \right] \\ & - U_1 (z - z^*)^2 \left[d'_1 - \frac{\alpha_1 p}{\sigma_2} (1 - e_1 p^*) - \frac{\alpha_2 f^*}{\sigma_3} (e_2 + z z^*) \right] - U_3 (e_T - e_T^*)^2 [d'_3 + \alpha_4 p] \\ & - U_4 \gamma'_2 (n - n^*)^2 - (p - p^*) (z' - z^*) \left[\frac{1 + e_1 p^* - \alpha_1 z^* U_1}{\sigma_2} \right] \\ & - (e_T - e_T^*) (n - n^*) [U_3 \alpha_5 p - \gamma'_2 U_4] \\ & - \frac{(z - z^*) (f - f^*)}{\sigma_3} [U_1 e_2 z' (z^{*2} + e_2) + U_2 \alpha_3 f^* (z z^* - \alpha_2)] \\ & - (p - p^*) (e_T - e_T^*) \left[\frac{\gamma\gamma_1 p^*}{\sigma_1} (1 - p) + \alpha_4 U_3 e_T^* + \alpha_5 U_3 n^* \right] \end{aligned} \quad (25)$$

where,

$$\sigma_1 = (1 + \gamma\gamma_1 p e_T)(1 + \gamma\gamma_1 p^*), \sigma_2 = (1 + e_1 p)(1 + e_1 p^*), \sigma_3 = (z^2 + e_2)(z^{*2} + e_2)$$

and choosing $U_1 = \frac{1 + e_1 p^*}{\alpha_1 z^*} > 0$ from above eq. (25). Now, $\frac{dV}{dt}$ can further be written as sum of the quadratic form as

$$\begin{aligned} \frac{dV}{dt} \leq & -[(a_{22}/2)(z - z^*)^2 + a_{23}(z - z^*)f - f^*] + (a_{33}/2)(f - f^*)^2 \\ & + ((a_{11}/2)(p - p^*)^2 + a_{14}(p - p^*)(e_T - e_T^*) + (a_{44}/2)(e_T - e_T^*)^2) \\ & + ((a_{44}/2)(e_T - e_T^*)^2 + a_{45}(e_T - e_T^*)(n - n^*) + (a_{55}/2)(n - n^*)^2) \end{aligned}$$

where,

$$\begin{aligned} a_{11} &= \frac{1 + \gamma\gamma_1 e_T}{\sigma_1} - \frac{e_1 z^*}{\sigma_2}, & a_{44} &= U_3 [d'_3 + \alpha_4 p] \\ a_{22} &= U_1 \left[d'_1 - \frac{\alpha_1 p}{\sigma_2} (1 - e_1 p^*) - \frac{\alpha_2 f^*}{\sigma_3} (e_2 + z z^*) \right] \\ a_{33} &= U_2 \left[d'_2 - \frac{\alpha_3 z}{\sigma_3} (z^{*2} + e_2) \right] \\ a_{23} &= \frac{1}{\sigma_3} [U_1 e_2 z (z^{*2} + e_2) + U_2 \alpha_3 f^* (z z^* - \alpha_2)] \\ a_{14} &= \left[\frac{\gamma\gamma_1 p^*}{\sigma_1} (1 - p) + \alpha_4 U_3 e_T^* + \alpha_5 U_3 n^* \right] \\ a_{45} &= U_3 \alpha_5 p - \gamma'_2 U_4, & a_{55} &= U_4 \gamma'_2 \end{aligned}$$

Now, by selecting $U_2 = U_3 = U_4 = 1$ and applying Sylvester's criteria, we can determine that $\frac{dV}{dt}$ is negative definite under the subsequent circumstances:

$$a_{11} > 0 \quad (26)$$

$$a_{22} > 0 \quad (27)$$

$$a_{33} > 0 \quad (28)$$

$$a_{11}a_{44} > a_{14}^2 \quad (29)$$

$$a_{22}a_{33} > a_{23}^2 \quad (30)$$

$$a_{44}a_{55} > a_{45}^2 \quad (31)$$

We note that the inequalities, (19) \Rightarrow (26), (20) \Rightarrow (27), (21) \Rightarrow (28), (22) \Rightarrow (29), (23) \Rightarrow (30) and (24) \Rightarrow (31). Therefore V is a Liapunov function considered for E_3^* , which is observed in the region of attraction Ω , validates the theorem.

5 Simulation and Discussion

For numerical simulation, we have used MATLAB and MATHEMATICA software. We illustrate the dynamic behavior of a tri-trophic food chain system influenced by the toxicants and distributed delay with the support of numerical examples.

We select the following parameter values for \hat{E}_1 :

$$\begin{aligned} r = 0.5, \quad k = 1, \quad \gamma = 0.1, \quad \gamma_1 = 0.1, \quad \beta = 0.1245, \quad \alpha = 0.0992, \quad \beta_1 = 0.144, \\ \beta_2 = 0.099, \quad d_1 = 0.8935, \quad \beta_3 = 0.38, \quad j = 0.05, \quad d_2 = 0.0225, \quad a = 0.11825, \quad A = 0.089, \\ a_1 = 0.08, \quad d_3 = 0.35, \quad \gamma_2 = 0.119. \end{aligned}$$

It has been determined that, under the above specific set of parameters, the equilibrium point $\hat{E}_1=(1,0,0,0.2069,0.2069)$ is locally asymptotically stable (see Fig. 1). If we increase the values of γ , oscillation occurs.

We select the following parameter values for \bar{E}_2 :

$$\begin{aligned} r = 0.48, \quad k = 4.15, \quad \gamma = 0.94, \quad \gamma_1 = 0.09, \quad \beta = 0.39, \quad \alpha = 1.899, \quad \beta_1 = 0.153, \\ \beta_2 = 0.0298, \quad d_1 = 0.1, \quad \beta_3 = 0.0259, \quad j = 0.5, \quad d_2 = 0.1, \quad a = 0.1185, \quad A = 0.289, \\ a_1 = 0.02, \quad d_3 = 0.5, \quad \gamma_2 = 1.38. \end{aligned}$$

It has been determined that, under the above set of parameters, the equilibrium point $\bar{E}_2=(0.8637,0.3858,0,0.3044,0.3044)$ is locally asymptotically stable (see Fig. 2). When we compare these values with \hat{E}_1 we get that phytoplankton decreases due to the presence of zooplankton and toxicant.

We select the following parameter values for E_3^* :

$$\begin{aligned} r = 1.45, \quad k = 1.19, \quad \gamma = 2.29, \quad \gamma_1 = 9.85, \quad \beta = 0.54, \quad \alpha = 0.49, \quad \beta_1 = 0.53, \\ \beta_2 = 0.52, \quad d_1 = 0.1, \quad \beta_3 = 0.51, \quad j = 1.42, \quad d_2 = 0.49, \quad a = 0.1256, \quad A = 0.1956, \\ a_1 = 0.1129, \quad d_3 = 1.95, \quad \gamma_2 = 0.1 \end{aligned}$$

It has been determined that for the considered parameters, $E_3^*=(0.6441,0.4861,0.04461,0.0622,0.0622)$ equilibrium point is locally asymptotically stable (see Fig. 3). When we compare these equilibrium values with \hat{E}_1 and \bar{E}_2 we observe that value of phytoplankton and zooplankton decrease due the presence of fish and toxicant.

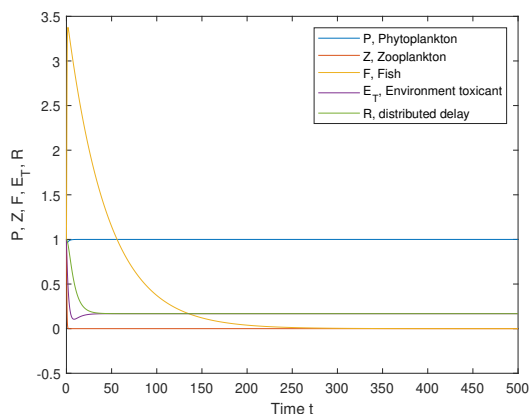


Figure 1: The time graph of the Model illustrates the stability behavior around equilibrium point \hat{E}_1 .

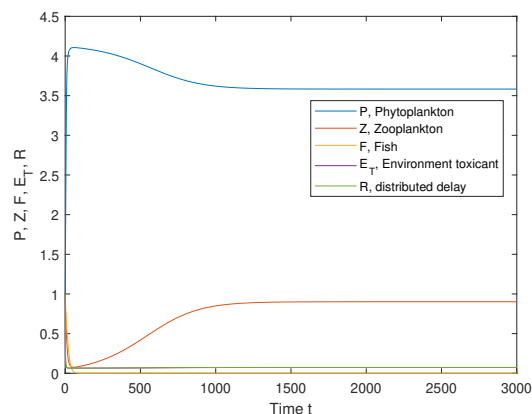


Figure 2: The time graph of the Model illustrates the stability behavior around equilibrium point \bar{E}_2 .

Now we check the bound for E_3^* we got that $0 \leq p \leq 1, 0 \leq 1.1265z + 1.6935f \leq K_1 = 14.51, 0 \leq e_T \leq K_2 = 0.1193, 0 \leq e_T + n \leq k_3 = 2.3298$ which proves that the system is bounded.

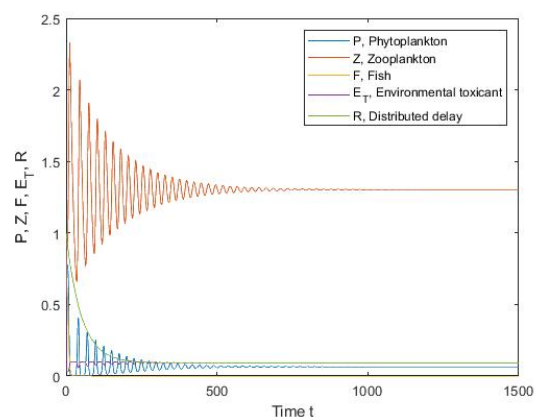


Figure 3: The model's time graph, at the equilibrium point where a stable graph is obtained.

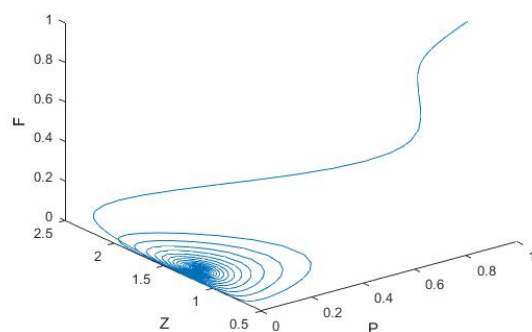


Figure 4: Time phase diagram of the Model near the equilibrium point where a stable graph is obtained.

The bifurcation diagram of the model for parameter γ is presented in Fig. 7 and Fig. 8.

$$\begin{aligned}
 r &= 1.45, & k &= 1.19, & \gamma &= 1.1, & \gamma_1 &= 4.1, & \beta &= 0.54, & \alpha &= 0.49, & \beta_1 &= 0.53, \\
 \beta_2 &= 0.52, & d_1 &= 0.06, & \beta_3 &= 0.51, & j &= 1.3, & d_2 &= 0.39, & a &= 0.1129, & A &= 0.1296, \\
 a_1 &= 0.1129, & d_3 &= 1.95, & \gamma_2 &= 0.01
 \end{aligned}$$

We study the Hopf-bifurcation taking γ as a bifurcating parameter. The transversality conditions

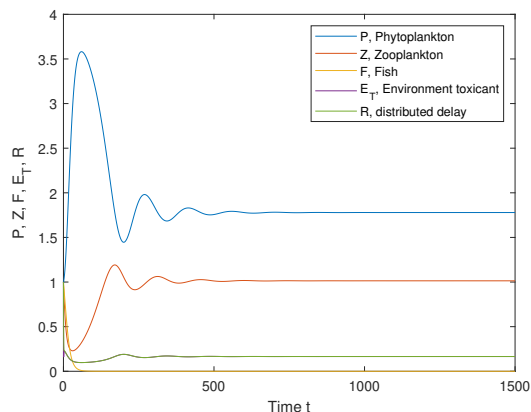


Figure 5: The time graph of the Model indicates stable behavior around equilibrium point E_3^* , however routh's conditions are not satisfied.

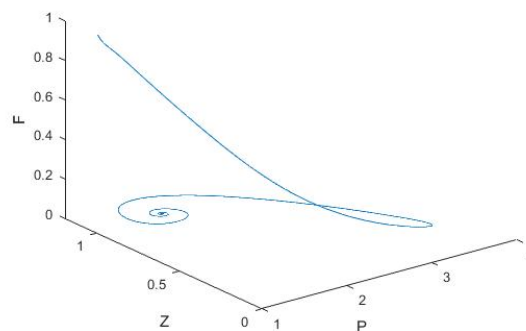


Figure 6: Time phase graph of the Model indicates stable behavior around equilibrium point E_3^* , however routh's conditions are not satisfied.

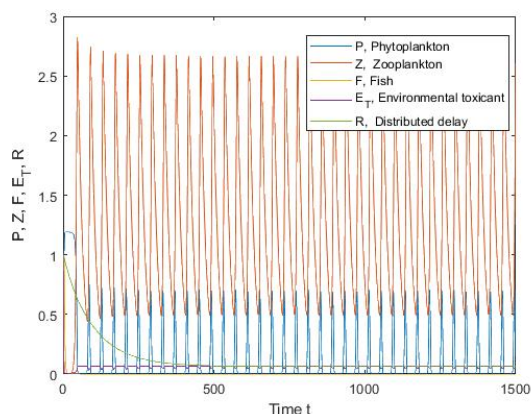


Figure 7: The bifurcation behavior is depicted in the Model's time graph around the equilibrium point E_3 at $\gamma = 1.1$.

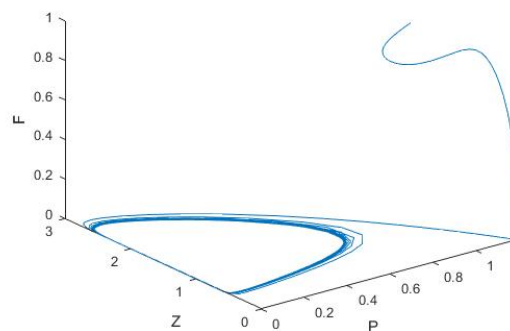


Figure 8: The bifurcation behavior is displayed in the Model's phase graph around the equilibrium point E_3 at $\gamma = 1.1$.

satisfy the above set of parameters when $\gamma = \bar{\gamma} = 5.9$. It is clear that interior equilibrium point E_3^* will be stable when $\gamma > \bar{\gamma}$ and unstable when $\gamma \leq \bar{\gamma}$ for which Hopf-bifurcation occurs.

6 Conclusion

In this study, we developed and examined a mathematical model to investigate the impact of toxicants and distributed delay in a tri-trophic food chain model. We have assumed in the model, the growth rate of phytoplankton in the absence of environmental toxin is in logistic form. We have incorporated Holling type II and III responses. The local stability study of all the equilibrium points is discussed. Global stability is studied for only nontrivial equilibrium points. It is concluded from the stability of $\hat{E}_1(\hat{p}, 0, 0, \hat{e}_T, \hat{n})$ that only the population of phytoplankton will survive and the rest will tend

to vanish (see Fig. 1). Through the stability of $\bar{E}_2(\bar{p}, \bar{z}, 0, \bar{e}_T, \bar{n})$, the population of phytoplankton and zooplankton will survive and fish may die out (see Fig. 2). In the case of interior equilibrium $E_3^*(p^*, z^*, f^*, e_T^*, n^*)$, we have a set of parameters for which the population density of zooplankton is higher than the density of phytoplankton (see Fig. 3). Also, there are specific parameters that allow for the simultaneous existence of all considered populations (see Fig. 5). These values reveal

Table 1: Numerical values of the model without toxicant and without delay

Equilibrium points	Numerical points
$\hat{E}_1(\hat{p}, 0, 0)$	(1,0,0)
$\bar{E}_2(\bar{p}, \bar{z}, 0)$	(0.8637,0.3943,0)
$E_3^*(p^*, z^*, f^*)$	(0.8401,0.4861,0.0159)

that the equilibrium density decreases due to the presence of environmental toxicants. In this study, Hopf bifurcation around its positive equilibrium point is examined with respect to γ which is the contact rate between phytoplankton and environmental toxin. The system remains stable until these parameters reach their critical values; beyond these points, the system becomes unstable. Furthermore, it is noted that at the equilibrium point \bar{E}_2 , the zooplankton population experiences a decline as a result of environmental effects and distributed delay whereas at the equilibrium E_3^* , phytoplankton population is diminished as a consequence of environmental toxicants and distributed delay (see Table 1 and Table 2). Therefore, it can be concluded that the effect of toxins on the aquatic ecosystem has a significant impact. [15].

Table 2: Numerical values of the model with toxicant and delay

Equilibrium points	Numerical points
$\hat{E}_1(\hat{p}, 0, 0, \hat{e}_T, \hat{n})$	(1,0,0,0.2069,0.2069)
$\bar{E}_2(\bar{p}, \bar{z}, 0, \bar{e}_T, \bar{n})$	(0.8637,0.3858,0,0.3044,0.3044)
$E_3^*(p^*, z^*, f^*, e_T^*, n^*)$	(0.7268,0.4861,0.0130,0.0586,0.0586)

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