A Generalized Class of Ratio Type Estimator of Population Variance based on Quartiles and Tri-means

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Abstract

This article examines the approximation of employing Quartile's tri mean and quartile range etc. of auxiliary information to estimate an unknown population variance. A generalized class of estimator has been proposed for estimating the population variance. The bias and the mean square error of the proposed estimator have been obtained up to the first order of approximations. The proposed estimator has been optimized with optimizing constant and minimum MSE found. The mean square error (MSE) of the proposed estimator have been compared theoretically and empirically and simulated data with the PRE of existing estimators. The proposed class of estimator outperform than many similar estimators, such as usual variance estimator, ratio estimator, product estimator in lines of efficiency.

Keyword: Bias, mean square square, Auxiliary variable, Variance estimation, Quartiles, Tri-mean

Introduction:

Auxiliary information is generally used to increase the efficiency and precision of estimators in sampling under various sampling strategies. To obtain an accurate estimate of the relevant population parameters, auxiliary information used which reduce the mean square error. Cochran (1940) was the first who introduced the use of auxiliary information in sampling to enhance the efficiency of the estimators. Taking into account the relationship between survey variable and auxiliary variables, a sufficient research has been done to improve the effectiveness of ratio and product estimators of finite population variance. Auxiliary variables are those that have a positive or negative correlation with the primary study variable. For instance, while calculating the variance in air quality, auxiliary variables like temperature or wind speed are taken into account. Employment rates may be used as an auxiliary variable in economic surveys to estimate the variance in household income. When evaluating variance in patient recovery times in healthcare planning, patient demographics or medical histories are used as auxiliary variables. When there is a positive correlation between study variable Y and auxiliary variable X and regression line Y on X passes through the origin, the Ratio type variance estimator is employed and when there is negative correlation between study variable Y and auxiliary variable X and regression line crosses through the origin, the Product type variance estimator is employed to increase the accuracy of the estimate under the random sampling. It is well known that the values of quartiles and their functions are unaffected by extreme values or the existence of outliers in the population therefor these parameters of auxiliary variable should be preferred to use in the estimator. Several authors examine the problem of estimating the population variance of the study variable using information on the variance, quartiles, inter-quartile range, semi-quartile range, and semi-quartile average of an auxiliary variable.

Many authors have addressed the issue of suggesting effective estimators for the population variance such as Isaki (1983) proposed variance estimator using auxiliary information, Wolter (1985) suggested estimator to improve variance estimator, Upadhyaya, L.N. and Singh, H.P. (2006) proposed, almost unbiased ratio and product type estimators of finite population variance in sample surveys, Kadilar and Cingi (2006) suggested an improvement in variance estimation using auxiliary information in simple random sampling, Subramani and Kumarapandiyan (2012a) proposed variance estimation using median of the auxiliary variable. Also Subramani & Kumarapandiyan (2012b) proposed variance estimators using quartiles and their functions of an auxiliary variable. Further Subramani & Kumarapandiyan (2013) proposed a new modified ratio estimator of population mean when median of the auxiliary variable is known, Khan & Shabbir (2013) proposed a ratio type estimator for the estimation of population variance using quartiles of an auxiliary variable. Maqbool and Javaid (2017) proposed variance estimator using linear combination of tri-mean and quartile average. Khalil et al. (2018) suggested an improved estimator of population variance using measure of dispersion of auxiliary variable. Yadav et al. (2019) proposed an efficient estimator of population variance using tri-mean and third quartile of auxiliary variable. Sharma et al. (2022) suggested an improvement in estimation of population variance utilising known auxiliary parameters for a decision making model. Shiv Shankar Soni and Pandey (2022) suggested an improved estimation of population variance utilizing known auxiliary parameters.

In this work, we proposed a class of estimators for variance that use variance, quartiles, and their functions of the auxiliary variable. The characteristics of the proposed class of estimators are studied under large sample approximation. The proposed class of estimators has been proven to be more efficient than related existing estimators both theoretically and empirically.

Some Notations:-

The Notations used in this paper are presented below

N = Population size, n = Sample size, $\gamma = \frac{1}{n}$ is a constant.

Y = Study variable, X = Auxiliary variable

$\overline{Y} = \frac{1}{N}\sum_{i=1}^{N} Y_i$ is population mean of study variable Y, $\overline{X} = \frac{1}{N}\sum_{i=1}^{N} X$ is population mean of auxiliary variable X.

$\overline{y} = \frac{1}{n}\sum_{j=1}^{n} y_j$ are sample mean of study variable y, $\overline{x} = \frac{1}{N}\sum_{j=1}^{n} x_j$ is sample mean of auxiliary variable x.

$S_y^2 = \frac{\sum_{i=1}^{N}(Y_i-\overline{Y})^2}{N-1}$ is population variance of study variable Y, $S_x^2 = \frac{\sum_{i=1}^{N}(X_i-\overline{X})^2}{N-1}$ is population variance of auxiliary variable X.

$s_y^2 = \sum_{j=1}^{n}\frac{\sum_{j=1}^{n}(y_i-\overline{y})^2}{n-1}$ is sample variance of study variable y, $s_x^2 = \frac{\sum_{j=1}^{n}(x_j-\overline{x})^2}{n-1}$ is sample variance of auxiliary variable x.

$C_y = \frac{S_y}{\overline{Y}}$ is coefficient of variation of y variable and $C_x = \frac{S_x}{\overline{Y}}$ is coefficient of variation of x variable.

$C_{yx} = \rho_{yx}C_yC_x$, ρ_{yx} = Corr. coeff. between y and x variables.

$\beta_{1(x)}$ = Skewness of the auxiliary x variable and $\beta_{2(x)}$ = Kurtosis of the auxiliary variable x.

M_d = Median of the auxiliary variable x

$$\lambda_{rs} = \frac{\mu_{rs}}{\mu_{02}^{\frac{r}{2}}\mu_{20}^{\frac{s}{2}}} \quad , \mu_{rs} = \frac{1}{N-1}\sum(Y_i-\overline{Y})^r(X_i-\overline{X})^s$$

Q_i is the ith quartile (i=1, 2, 3) of the auxiliary variable x.

$Q_d = \frac{(Q_3 - Q_1)}{2}$ is population quartile deviation of auxiliary variable x.

$Q_a = \frac{(Q_3 + Q_1)}{2}$ is population semi quartile average of auxiliary variable x.

TM = Tri mean = $\frac{(Q_1 + 2Q_2 + Q_3)}{4}$

Some Existing Estimators: -

The most suitable estimator of population variance S_y^2 is the sample variance s_y^2 , given by,

$$t_0 = s_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2 \quad (1)$$

The variance of t_0 for an approximation of degree one is,

$$V(t_0) = \gamma\, S_y^4(\lambda_{40} - 1) \quad (2)$$

Isaki (1983) suggested the classical ratio estimator for population variance is given as,

$$t_1 = s_y^2\left(\frac{S_x^2}{s_x^2}\right) \quad (3)$$

It is biased estimator and its MSE is given as,

$$MSE(t_1) = \gamma\, S_y^4[(\lambda_{40} - 1) + (\lambda_{04} - 1) - 2(\lambda_{22} - 1)] \quad (4)$$

Upadhyaya and Singh (1999) introduced a ratio type estimator of population variance based on kurtosis of auxiliary variable which is given as,

$$t_2 = s_y^2\left(\frac{S_x^2 + \beta_{2(x)}}{s_x^2 + \beta_{2(x)}}\right) \quad (5)$$

The MSE expression is given as,

$$MSE(t_1) = \gamma\, S_y^4[(\lambda_{40} - 1) + R_1^2(\lambda_{04} - 1) - 2R_1(\lambda_{22} - 1)] \quad (6)$$

Where, $R_1 = \frac{S_x^2}{S_x^2 + \beta_{2(x)}}$

Kadilar and Cingi (2006) introduced some estimator of population variance based on C_x , $\beta_{2(x)}$ there are given as,

$$t_i = s_y^2\left(\frac{a_i S_x^2 + b_i}{a_i s_x^2 + b_i}\right),\ \text{for}\ a_1 = 1, b_1 = C_x\ a_2 = \beta_{2(x)}\ b_2 = C_x\ , a_3 = C_x\ b_3 = \beta_{2(x)} \quad (7)$$

The MSEs of t_i $(i = 3,4,5)$ of first order approximation are given as,

$$MSE(t_i) = \gamma\, S_y^4[(\lambda_{40} - 1) + R_i^2(\lambda_{04} - 1) - 2R_i(\lambda_{22} - 1)] \quad (8)$$

Where , $R_2 = \frac{S_x^2}{S_x^2 + C_x}$ $\quad R_3 = \frac{S_x^2 \beta_{2(x)}}{S_x^2 \beta_{2(x)} + C_x}$ $\quad R_4 = \frac{S_x^2 C_x}{S_x^2 C_x + \beta_{2(x)}}$

Subramani and Kumarpandiyan (2012a) suggested an estimator of population variance by using the median M_d of auxiliary variable X is given as,

$$t_6 = s_y^2 \left(\frac{S_x^2+M_d}{s_x^2+M_d}\right) \quad (9)$$

The MSE expression is given as,

$$MSE(t_6) = \gamma\, S_y^4[(\lambda_{40}-1) + R_5^2(\lambda_{04}-1) - 2R_5(\lambda_{22}-1)] \quad (10)$$

Where, $R_5 = \frac{S_x^2}{S_x^2+M_d}$

Subramani and Kumarpandiyan (2012b) suggested some estimator of population variance by using the quartiles of X and their functions of X is given as,

$$t_i = s_y^2 \left(\frac{a_i S_x^2+b_i}{a_i s_x^2+b_i}\right), \text{for } a_i = 1,\ b_1 = Q_1, b_2 = Q_3, b_3 = Q_r, b_4 = Q_d, b_5 = Q_a \quad (11)$$

The MSEs of t_i $(i = 7,8,9,10,11)$ of first order approximation are given as,

$$MSE(t_i) = \gamma\, S_y^4[(\lambda_{40}-1) + R_i^2(\lambda_{04}-1) - 2R_i(\lambda_{22}-1)] \quad (12)$$

Where, $R_6 = \frac{S_x^2}{S_x^2+Q_1}$, $R_7 = \frac{S_x^2}{S_x^2+Q_3}$, $R_8 = \frac{S_x^2}{S_x^2+Q_r}$, $R_9 = \frac{S_x^2}{S_x^2+Q_d}$, $R_{10} = \frac{S_x^2}{S_x^2+Q_a}$,

Subramani and Kumarpandiyan (2013) suggested an estimator of population variance by using the C_x and M_d is given as,

$$t_{12} = s_y^2 \left(\frac{S_x^2 C_x+M_d}{s_x^2 C_x+M_d}\right) \quad (13)$$

The MSE of above estimator is given as,

$$MSE(t_{12}) = \gamma\, S_y^4[(\lambda_{40}-1) + R_{12}^2(\lambda_{04}-1) - 2R_{12}(\lambda_{22}-1)] \quad (14)$$

Where, $R_{11} = \frac{S_x^2 C_x}{S_x^2 C_x+M_d}$

Khan and Shabbir (2013) introduced an estimator of population variance by using the correlation coefficient and third quartile of X, which is given as:

$$t_{13} = s_y^2 \left(\frac{S_x^2\rho+Q_3}{s_x^2\rho+Q_3}\right) \quad (15)$$

The MSE of above estimator is given as,

$$MSE(t_{13}) = \gamma\, S_y^4[(\lambda_{40}-1) + R_{12}^2(\lambda_{04}-1) - 2R_{12}(\lambda_{22}-1)] \quad (16)$$

Where, $R_{12} = \frac{S_x^2\rho}{S_x^2\rho+Q_3}$

Bhat Maqbool and Raja (2017) introduced a new modified ratio type variance estimator of auxiliary variable by using linear combination of Tri mean and Quartiles which are given as,

$$t_i = s_y^2 \left(\frac{a_i S_x^2+b_i}{a_i s_x^2+b_i}\right) \quad (17)$$

for $a_i = 1, b_1 = TM + Q_1, b_2 = TM + Q_2, b_3 = TM + Q_3, b_4 = TM + Q_d, b_5 = TM + Q_a,$ $b_5 = TM + Q_r$

The MSEs of t_i $(i = 14 \dots \dots \dots 19)$ of first order approximation are given as,

$$MSE(t_i) = \gamma\, S_y^4[(\lambda_{40} - 1) + R_i^2(\lambda_{04} - 1) - 2R_i(\lambda_{22} - 1)] \quad (18)$$

Where, $R_{13} = \frac{S_x^2}{S_x^2 + (TM + Q_1)}$, $R_{14} = \frac{S_x^2}{S_x^2 + (TM + Q_2)}$, $R_{15} = \frac{S_x^2}{S_x^2 + (TM + Q_3)}$,

$R_{16} = \frac{S_x^2}{S_x^2 + (TM + Q_d)}$, $R_{17} = \frac{S_x^2}{S_x^2 + (TM + Q_a)}$, $R_{18} = \frac{S_x^2}{S_x^2 + (TM + Q_r)}$

Shahzad et al. (2018) suggested an improved ratio type variance estimator by using linear combination of different measures of location, given as

$$t_{20} = s_y^2\left(\frac{a_i S_x^2 + b_i}{a_i s_x^2 + b_i}\right) \quad (19)$$

for $a_i = \rho$, $b_1 = M_d + Q_1$, $b_2 = M_d + Q_2$, $b_3 = M_d + Q_3$, $b_4 = M_d + Q_d$, $b_4 = M_d + Q_r$, $b_6 = M_d + Q_a$

The MSEs of t_i $(i = 20 \dots \dots \dots 25)$ of first order approximation are given as,

$$MSE(t_i) = \gamma\, S_y^4[(\lambda_{40} - 1) + R_i^2(\lambda_{04} - 1) - 2R_i(\lambda_{22} - 1)] \quad (20)$$

Where, $R_{19} = \frac{S_x^2}{S_x^2 + (TM + Q_1)}$, $R_{20} = \frac{S_x^2}{S_x^2 + (TM + Q_2)}$, $R_{21} = \frac{S_x^2}{S_x^2 + (TM + Q_3)}$,

$R_{22} = \frac{S_x^2}{S_x^2 + (TM + Q_d)}$, $R_{23} = \frac{S_x^2}{S_x^2 + (TM + Q_a)}$, $R_{24} = \frac{S_x^2}{S_x^2 + (TM + Q_r)}$

Sharma et al. (2022) introduced an estimator of population variance based on TM and Q_3 are given as,

$$t_{26} = k s_y^2\left(\frac{S_x^2 + (TM + Q_3)}{s_x^2 + (TM + Q_3)}\right) \quad (21)$$

The MSE of above estimator is given as,

$$MSE(t_{26}) = S_y^4[1 + k^2 B_1 - 2B_2] \quad (22)$$

Where,

$B_1 = 1 + \gamma(\lambda_{40} - 1) + 3R_{25}^2\gamma(\lambda_{04} - 1) - 4R_{25}\gamma(\lambda_{22} - 1)$

$B_2 = 1 + R_{25}^2\gamma(\lambda_{04} - 1) - R_{25}\gamma(\lambda_{22} - 1)$, $R_{25} = \frac{S_x^2}{S_x^2 + (TM + Q_3)}$

Shiv Shankar Soni & Himanshu Pandey (2022) suggested a class of estimator by using TM & Q_3 is given as,

$$t_{27} = k_1 s_y^2 + k_2 s_y^2\left[\frac{S_x^2 + (TM + Q_3)}{s_x^2 + (TM + Q_3)}\right] \quad (23)$$

The MSE of above estimator of first order approximation is given as

$$MSE(t_{27}) = S_y^4[1 + k_1^2 A + k_2^2 B - 2k_1 - 2k_2 C + 2k_1 k_2 D] \quad (24)$$

Where,

$A = \{1 + \gamma(\lambda_{40} - 1)\}$, $B = \{1 + \gamma(\lambda_{40} - 1) + 3R_{25}^2\gamma(\lambda_{04} - 1) - 4R_{25}\gamma(\lambda_{22} - 1)\}$

$C = \{1 + R_{25}^2\gamma(\lambda_{04} - 1) - R_{25}\gamma(\lambda_{22} - 1)\}$

$D = \{1 + \gamma(\lambda_{40} - 1) + R_{25}^2\gamma(\lambda_{04} - 1) - 2R_{25}\gamma(\lambda_{22} - 1)\}$

Proposed Estimator

Motivated by Sharma et al.(2022) we suggested a class of estimator of population variance S_y^2 using on some function of auxiliary variable which takes the form:

$$t_{ab1} = ks_y^2\left[\frac{aS_x^2+b}{as_x^2+b}\right] \tag{25}$$

Further motivated by Shiv Shankar Soni (2022) we suggestesd another estimator

$$t_{ab2} = k_1s_y^2 + k_2s_y^2\left[\frac{aS_x^2+b}{as_x^2+b}\right] \tag{26}$$

Where a and b are constant are based on parameters.

As we know that,

$e_o = \frac{s_y^2}{S_y^2} - S_y^2$ and $e_1 = \frac{s_x^2}{S_x^2} - S_x^2$ such that,

$E(e_o) = E(e_1) = 0$

$E(e_o^2) = \gamma(\lambda_{40} - 1)$

$E(e_1^2) = \gamma(\lambda_{04} - 1)$

$E(e_oe_1) = \gamma(\lambda_{22} - 1)$

Where $\gamma = \frac{1}{n}, \lambda_{rs} = \frac{\mu_{rs}}{\mu_{02}^{\frac{s}{2}}\mu_{20}^{\frac{r}{2}}}$, $\mu_{rs} = \frac{1}{N-1}\sum(Y_i - \overline{Y})^r(X_i - \overline{X})^s$ and r & s any non-negative integer.

The proposed estimator may be expressed in terms of e's as,

$$t_{ab1} = kS_y^2(1 + e_0)(1 + R_{26}e_1 + R_{26}^2e_1^2) \tag{27}$$

$$t_{ab1} = kS_y^2(1 + e_0 - R_{26}e_1 + R_{26}^2e_1^2 - R_{26}e_0e_1) \tag{28}$$

$$t_{ab1} - S_y^2 = S_y^2[(k-1) + k(e_0 - R_{26}e_1 + R_{20}^2e_1^2 - R_{26}e_0e_1)] \tag{29}$$

Taking expectation on both side and taking up to first order of approximation, we get bias

$$Bias(t_{ab1}) = S_y^2\left[(k-1) + k\left(R_{26}^2\gamma(\lambda_{04} - 1) - R_{26}\gamma(\lambda_{22} - 1)\right)\right] \tag{30}$$

Squaring both side of eq. and taking expectation both side, we get the MSE of proposed estimator,

$$\mathrm{MSE}(t_{ab1}) = S_y^4[1 + k^2(1 + \gamma(\lambda_{40} - 1) + 3R_{26}^2\gamma(\lambda_{04} - 1) - 4R_{26}\gamma(\lambda_{22} - 1) - 2K(1 + R_{26}^2\gamma(\lambda_{04} - 1) - R_{26}\gamma(\lambda_{22} - 1))] \quad (31)$$

Or $$\mathrm{MSE}(t_{ab1}) = S_y^4[1 + k^2B_1 - 2B_2] \quad (32)$$

After differentiating with respect to k we get the optimum value of k

$$k^*_{opt.} = \left[\frac{B_2}{B_1}\right]$$

Using the value of $k^*_{opt.}$ we get the minimum value of MSE which is

$$\mathrm{Min.\,MSE}(t_{ab1}) = S_y^4\left[1 - \frac{B_2^2}{B_1}\right] \quad (33)$$

Where,

$$B_1 = 1 + \gamma(\lambda_{40} - 1) + 3R_{26}^2\gamma(\lambda_{04} - 1) - 4R_{26}\gamma(\lambda_{22} - 1)$$

$$B_2 = 1 + R_{26}^2\gamma(\lambda_{04} - 1) - R_{26}\gamma(\lambda_{22} - 1) \quad R_{26} = \frac{aS_x^2}{aS_x^2+b}$$

Similarly, another proposed estimator t_{ab2} may be expressed as,

$$t_{ab2} = k_1s_y^2(1 + e_0) + k_2S_y^2(1 + e_0)(1 + R_{26}e_1 + R_{26}^2e_1^2) \quad (34)$$

$$t_{ab2} = k_1S_y^2(1 + e_0) + k_2S_y^2(1 + e_0 - R_{26}e_1 + {R_{26}}^2e_1^2 - R_{26}e_0e_1) \quad (35)$$

$$t_{ab2} - S_y^2 = S_y^2\left[k_1(1 + e_0) + k_2\left(1 + e_0 - R_{26}e_1 + {R_{26}}^2e_1^2 - R_{26}e_0e_1\right)\right] \quad (36)$$

Taking expectation on both side and taking up to first order of approximation, we get bias

$$\mathrm{Bias}(t_{ab2}) = S_y^2\left[(k_1 - 1) + k_2\left(1 + {R_{26}}^2\gamma(\lambda_{04} - 1) - R_{26}\gamma(\lambda_{22} - 1)\right)\right] \quad (37)$$

Squaring both side of eq. (37) and taking expectation both side, we get the MSE of proposed estimator,

$$\mathrm{MSE}(t_{ab2}) = S_y^4[1 + k_1^2A_1 + k_2^2A_2 - 2k_1 - 2k_2A_3 + 2k_1k_2A_4] \quad (38)$$

After differentiating with respect to k_1 and k_2 we get the optimum value of it,

$$k_1^* = \frac{A_4A_3 - A_2}{A_4^2 - A_1A_2} \text{ and } k_2^* = \frac{A_4 - A_1A_3}{A_4^2 - A_1A_2}$$

Using the value of k_1^* & k_2^* , we get the minimum value of MSE which are

$$\mathrm{Min.\,MSE}(t_{ab2}) = S_y^4\left[1 + \frac{A_1A_3^2 + A_2 - 2A_3A_4}{A_4^2 - A_1A_2}\right] \quad (39)$$

Where,

$A_1 = \{1 + \gamma(\lambda_{40} - 1)\}$,

$A_2 = \{1 + \gamma(\lambda_{40} - 1) + 3R_{26}^2\gamma(\lambda_{04} - 1) - 4R_{26}\gamma(\lambda_{22} - 1)\}$

$A_3 = \{1 + R_{26}^2\gamma(\lambda_{04} - 1) - R_{26}\gamma(\lambda_{22} - 1)\}$,

$A_4 = \{1 + \gamma(\lambda_{40} - 1) + R_{26}^2\gamma(\lambda_{04} - 1) - 2R_{26}\gamma(\lambda_{22} - 1)\}$

Table 1: Some members of the proposed generalized class of estimator (t_{ab1})

S. No	Estimators	Value of Constant		
		k	a	b
1.	$t_{14(1)} = s_y^2\left(\frac{\rho S_x^2+(M_d+Q_1)}{\rho s_x^2+(M_d+Q_1)}\right)$ Shahzad et al. (2018)	1	ρ	$M_d + Q_1$
2.	$t_{14(2)} = s_y^2\left(\frac{\rho S_x^2+(M_d+Q_2)}{\rho s_x^2+(M_d+Q_2)}\right)$ Shahzad et al. (2018)	1	ρ	$M_d + Q_2$
3.	$t_{14(3)} = s_y^2\left(\frac{\rho S_x^2+(M_d+Q_3)}{\rho s_x^2+(M_d+Q_3)}\right)$ Shahzad et al. (2018)	1	ρ	$M_d + Q_3$
4.	$t_{14(4)} = s_y^2\left(\frac{\rho S_x^2+(M_d+Q_d)}{\rho s_x^2+(M_d+Q_d)}\right)$ Shahzad et al. (2018)	1	ρ	$M_d + Q_d$
5.	$t_{14(5)} = s_y^2\left(\frac{\rho S_x^2+(M_d+Q_r)}{\rho s_x^2+(M_d+Q_r)}\right)$ Shahzad et al. (2018)	1	ρ	$M_d + Q_r$
6.	$t_{14(6)} = s_y^2\left(\frac{\rho S_x^2+(M_d+Q_a)}{\rho s_x^2+(M_d+Q_a)}\right)$ Shahzad et al. (2018)	1	ρ	$M_d + Q_a$
7.	$t_{20} = ks_y^2\left(\frac{S_x^2+(TM+Q_3)}{s_x^2+(TM+Q_3)}\right)$ Sharma et al. (2022)	k	1	$TM + Q_3$

Table 2: Some members of the proposed generalized class of estimator (t_{ab2})

S. No	Estimators	Value of Constant			
		k_1	k_2	a	b
1.	$t_{14} = s_y^2\left(\frac{S_x^2+(TM+Q_1)}{s_x^2+(TM+Q_1)}\right)$ Bhat and Maqbool(2017)	0	1	1	$TM + Q_1$
2.	$t_{15} = s_y^2\left(\frac{S_x^2+(TM+Q_2)}{s_x^2+(TM+Q_2)}\right)$	0	1	1	$TM + Q_2$

	Bhat and Maqbool(2017)				
3.	$t_{16} = s_y^2 \left(\frac{S_x^2+(TM+Q_3)}{s_x^2+(TM+Q_3)}\right)$ Bhat and Maqbool(2017)	0	1	1	$TM + Q_3$
4.	$t_{17} = s_y^2 \left(\frac{S_x^2+(TM+Q_d)}{s_x^2+(TM+Q_d)}\right)$ Bhat and Maqbool(2017)	0	1	1	$TM + Q_d$
6.	$t_{18} = s_y^2 \left(\frac{S_x^2+(TM+Q_a)}{s_x^2+(TM+Q_a)}\right)$ Bhat and Maqbool(2017)	0	1	1	$TM + Q_a$
7.	$t_{19} = s_y^2 \left(\frac{S_x^2+(TM+Q_r)}{s_x^2+(TM+Q_r)}\right)$ Bhat and Maqbool(2017)	0	1	1	$TM + Q_r$
9.	$t_{21} = k_1 s_y^2 + k_2 s_y^2 \left[\frac{S_x^2+(TM+Q_3)}{s_x^2+(TM+Q_3)}\right]$ Shiv Shankar Soni & Pandey (2022)	k_1	k_2	1	$TM + Q_3$

Efficiency Comparison

I. The estimator t_{ab1} is more efficient than $\bar{y}$ if $Var(S_y^2) - min. MSE(t_{ab1}) \geq 0$

$$S_y^4 \left[\gamma(\lambda_{40} - 1) - \left[1 - \frac{B_2^2}{B_1}\right]\right] \geq 0 \quad (40)$$

II. The estimator t_{ab1} is more efficient than t_1 if $MSE(t_1) - min. MSE(t_{ab1}) \geq 0$

$$S_y^4 \left[\gamma\{(\lambda_{40} - 1) + R_1^2(\lambda_{40} - 1) - 2R_1(\lambda_{22} - 1)\} - \left[1 - \frac{B_2^2}{B_1}\right]\right] \geq 0 \quad (41)$$

III. The estimator t_{ab1} is more efficient than t_i (i = 1,2,….19) if $MSE(t_i) - min. MSE(t_{ab1}) \geq 0$

$$S_y^4 \left[\gamma\{(\lambda_{40} - 1) + R_i^2(\lambda_{40} - 1) - 2R_i(\lambda_{22} - 1)\} - \left[1 - \frac{B_2^2}{B_1}\right]\right] \geq 0 \quad (42)$$

IV. The estimator t_{ab1} is more efficient than t_{20} if $MSE(t_{20}) - min. MSE(t_{ab1}) \geq 0$

$$S_y^4 \left[[1 + k^2B_1 - 2B_2] - \left[1 - \frac{B_2^2}{B_1}\right]\right] \geq 0 \quad (43)$$

V. The estimator t_{ab2} is more efficient than $\bar{y}$ if $Var(S_y^2) - min. MSE(t_{ab2}) \geq 0$

$$S_y^4 \left[\gamma(\lambda_{40} - 1) - S_y^4 \left[1 + \frac{A_1A_3^2+A_2-2A_3A_4}{A_4^2-A_1A_2}\right]\right] \geq 0 \quad (44)$$

VI. The estimator t_{ab2} is more efficient than t_1 if $MSE(t_1) - min. MSE(t_{ab2}) \geq 0$

$$S_y^4 \left[\gamma\{(\lambda_{40} - 1) + R_1^2(\lambda_{40} - 1) - 2R_1(\lambda_{22} - 1)\} - S_y^4 \left[1 + \frac{A_1A_3^2+A_2-2A_3A_4}{A_4^2-A_1A_2}\right]\right] \geq 0 \quad (45)$$

VII. The estimator t_{ab2} is more efficient than t_i (i = 1,2,3..19) if $MSE(t_i) - \min. MSE(t_{ab2}) \geq 0$

$$S_y^4\left[\gamma\{(\lambda_{40}-1)+R_i^2(\lambda_{40}-1)-2R_i(\lambda_{22}-1)\}-S_y^4\left[1+\frac{A_1A_3^2+A_2-2A_3A_4}{A_4^2-A_1A_2}\right]\right]\geq 0 \quad (46)$$

VIII. The estimator t_{ab2} is more efficient than t_{20} if $MSE(t_{20}) - \min. MSE(t_{ab2}) \geq 0$

$$S_y^4\left[[1+k^2B_1-2B_2]-S_y^4\left[1+\frac{A_1A_3^2+A_2-2A_3A_4}{A_4^2-A_1A_2}\right]\right]\geq 0 \quad (47)$$

Empirical Study

For the purpose of empirical study, we have considered two data set to examine the performance of the proposed estimator for the population variance of the study variable.

Population 1: [Source: Murthy (1967), p.228]

Y**:** Output of 80 factories in a region X**:** Fixed Capital

N = 80, n = 20, $\bar{Y} = 51.8264$, $\bar{X} = 11.2646$, $\rho = 0.9413$, $S_y = 18.3549$, $C_y = 0.3542$, $S_x = 8.4563$, $C_x = 0.7507$, $\lambda_{04} = 2.8664$, $\lambda_{40} = 2.2667$, $\lambda_{22} = 2.2209$, $Q_1 = 5.1500$, $Q_2 = M_d = 7.575$, $Q_3 = 16.975$, $Q_r = 11.825$, $Q_d = 5.9125$, $Q_a = 11.0625$ $T_m = 9.318$

Population 2: Govt. of Pakistan Statistics Division Federal Bureau of Statistics (Economics Wing) Islamabad (Area & Production of food crops in Punjab, Area in '000' Hec.

Y**:** Production of food crops in Punjab X**:** Area of food crops in Punjab

N = 33, n = 7, $\bar{Y} = 37.92$, $\bar{X} = 46.37$, $\rho = 0.9773$, $S_y = 14.87$, $C_y = 0.3922$, $S_x = 25.4$, $C_x = 0.5278$, $\lambda_{04} = 8.2369$, $\lambda_{40} = 8.282$, $\lambda_{22} = 5.51514$, $Q_1 = 33.9$, $Q_2 = M_d = 38.4$, $Q_3 = 47.7$, $Q_r = 40.8$, $Q_d = 6.9$, $Q_a = 81.6$ $T_m = 39.6$

Table 3: PREs of suggested estimators and some existing estimators for real data sets 1& 2

Estimators	PREs of Pop. 1	PREs of Pop. 2
Usual estimator (t_0)	100	100
Isaki (1983) (t_1)	137.42	131
Upadhyaya & Singh (1999) (t_2)	147.44	132
Kadilar & Cingi (2006) (t_3)	140.10	131
Kadilar & Cingi (2006) (t_4)	138.36	132
Kadilar & Cingi (2006) (t_5)	150.63	133
Subramani & Kumarpandiyan (2012a) (t_6)	129.72	139
Subramani & Kumarpandiyan (2012a) (t_7)	154.97	138
Subramani & Kumarpandiyan (2012a) (t_8)	185.44	141
Subramani & Kumarpandiyan (2012a) (t_9)	174.09	139

Subramani & Kumarpandiyan (2012a) (t_{10})	157.39	133
Subramani & Kumarpandiyan (2012a) (t_{11})	172.15	146
Subramani & Kumarpandiyan (2012a) (t_{12})	218.56	145
Khan & Shabbir (2013) (t_{13})	187.38	140
Bhat and Maqbool (2017) (t_{14})	240	144
Bhat and Maqbool (2017) (t_{15})	247	145
Bhat and Maqbool (2017) (t_{16})	264	147
Bhat and Maqbool (2017) (t_{17})	242	140
Bhat and Maqbool (2017) (t_{18})	256	150
Bhat and Maqbool (2017) (t_{19})	254	146
Sharma et al. (2022) (t_{20})	274	347
Shiv Shankar Soni (2022) (t_{21})	280	350

Table 4: PRE of Proposed Estimator t_{ab1}

	a	b	PRE 1	PRE 2		a	b	PRE 1	PRE 2
t_{ab1}	1	$TM + Q_1$	255	356	t_{ab1}	ρ	$TM + Q_d$	259	375
	1	$TM + Q_2$	260	353		ρ	$TM + Q_r$	270	351
	1	$TM + Q_3$	274	347		ρ	$TM + Q_a$	269	330
	1	$TM + Q_d$	257	376		ρ	$M_d + Q_1$	253	356
	1	$TM + Q_r$	268	352		ρ	$M_d + Q_2$	259	353
	1	$TM + Q_a$	267	331		ρ	$M_d + Q_3$	274	347
	ρ	$TM + Q_1$	257	355		ρ	$M_d + Q_d$	255	**376**
	ρ	$TM + Q_2$	263	352		ρ	$M_d + Q_r$	267	352
	ρ	$TM + Q_3$	**275**	347		ρ	$M_d + Q_a$	266	330

Table 5: PRE of Proposed Estimator t_{ab2}

	a	b	PRE 1	PRE 2		a	b	PRE 1	PRE 2
	1	$TM + Q_1$	284	358		Q_d	$TM + Q_1$	292	412
	1	$TM + Q_2$	283	355		Q_r	$TM + Q_1$	**293**	423
	1	$TM + Q_3$	280	350		Q_a	$TM + Q_1$	291	424
	1	$TM + Q_d$	284	378		Q_1	$TM + Q_d$	291	423

t_{ab2}	1	$TM + Q_a$	282	332	t_{ab2}	Q_2	$TM + Q_d$	292	423
	1	$TM + Q_r$	281	354		Q_3	$TM + Q_d$	**293**	**424**
	Q_1	$TM + Q_3$	290	422		Q_d	$TM + Q_d$	292	417
	Q_2	$TM + Q_3$	291	422		Q_r	$TM + Q_d$	293	423
	Q_3	$TM + Q_3$	**293**	423		Q_a	$TM + Q_d$	**293**	424
	Q_d	$TM + Q_3$	290	409		Q_1	$TM + Q_a$	291	420
	Q_a	$TM + Q_3$	292	**424**		Q_2	$TM + Q_a$	292	421
	Q_r	$TM + Q_3$	292	422		Q_3	$TM + Q_a$	**293**	422
	Q_1	$TM + Q_2$	291	422		Q_d	$TM + Q_a$	291	404
	Q_2	$TM + Q_2$	292	422		Q_r	$TM + Q_a$	293	421
	Q_3	$TM + Q_2$	**293**	423		Q_a	$TM + Q_a$	293	423
	Q_d	$TM + Q_2$	292	411		Q_1	$TM + Q_r$	291	422
	Q_a	$TM + Q_2$	**293**	**424**		Q_2	$TM + Q_r$	292	422
	Q_r	$TM + Q_2$	293	422		Q_3	$TM + Q_r$	**293**	423
	Q_1	$TM + Q_1$	292	422		Q_d	$TM + Q_r$	291	410
	Q_2	$TM + Q_1$	**293**	422		Q_r	$TM + Q_r$	292	422
	Q_3	$TM + Q_1$	**293**	423		Q_a	$TM + Q_r$	**293**	**424**

Simulation Study:

The effectiveness of the suggested estimators is evaluated by a simulation study. The PRE of the suggested estimators are calculated by using the following steps.

1. Generated two independent population of size 1000 by using the normal distribution such that $X = N(10,4)$ and $Z = N(10,9)$.
2. Set $Y = \rho X + \sqrt{(1-\rho^2)}Z$ where ρ is the correlation between X and Y.
3. The different values of correlation coefficient are taken as $\rho = 0.75,\ 0.85,\ 0.95$.
4. Select 10000 samples of size $n = 200, 300, 400$ respectively and compute the estimator for each samples.
5. The MSEs for each sample size for each estimator are obtained by using following formula

$$MSE(tab_j) = \frac{1}{10{,}000}\sum_{j=1}^{10{,}000}\left(tab_j - S_y^2\right)^2$$

6. The percent relative efficiency is calculated as

$$PRE(tab) = \frac{MSE(S_y^2)}{MSE(tab_j)} * 100$$

Table 6: PRE of suggested and some existing estimators' relatives to usual estimators at different sample size and different correlation coefficients.

t5	t6	t7	t8	t9	t10	t11	t12	t13	t14	t15	t16	t17	t18	t19	t20	t21	tab 1	tab 2
556	547	557	511	544	558	545	519	471	514	478	556	513	474	473	435	557	474	560
470	490	477	478	490	479	490	482	454	480	459	470	479	456	454	427	489	455	491
527	506	526	466	498	524	504	477	428	475	438	527	468	434	428	397	526	430	529
596	575	596	529	566	594	573	540	505	538	492	596	530	490	482	445	592	506	597
483	488	485	467	486	486	487	471	453	470	444	483	468	441	439	410	489	454	491
476	485	481	466	480	480	484	466	452	466	441	476	467	438	440	407	484	453	486
470	480	473	463	473	473	479	466	460	465	442	470	464	439	438	409	477	461	481
501	484	500	445	500	500	479	456	440	451	420	501	448	415	412	381	502	443	505
372	364	372	348	373	373	365	343	345	347	326	372	347	327	328	308	372	344	375

rho	n	t0	t1	t2	t3	t4
0.75	**200**	100	531	552	537	533
	300	100	433	460	440	435
	400	100	518	527	522	519
0.85	**200**	100	579	594	584	581
	300	100	452	476	458	454
	400	100	441	467	448	444
0.95	**200**	100	434	460	441	436
	300	100	493	501	496	494
	400	100	362	370	365	363

Conclusion:

In this paper, proposed a generalized class of ratio type variance estimator of population variance of the study variable by using known value of some parameters of auxiliary variable. The usual estimator Isaki (1983) Bhat and Maqbool (2017), Sharma et al. (2022) Shiv-Shankar Soni (2022) are particular cases of proposed generalized class of estimator. The Bias and mean square error of the proposed estimators are determined to the first order of approximation. Further the proposed estimators are efficient also in terms of PRE.

From above table 4 and table 5 we observed that PRE of the proposed estimators is higher than the PRE of the existing estimators. Therefore, proposed estimator performed not only generalized but also better than existing estimator. These estimators might be preferred over the previously suggested estimators in real world.

Further, from table 6 also shows the superiority over the existing estimators for different values of sample size i.e n = 200, 300, 400 and different values of correlation coefficient i.e ρ = 0.75,0.85, 0.95 .

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