Stability Analysis of an SEIE Epidemic Model with Non-Monotonic Saturated Incidence Rate

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Abstract

The present paper deals with a Susceptible-Exposed-Infected- Exposed epidemiological model with non-monotonic saturated incidence rate . Analyzed the stability for disease-free and endemic equilibria with a non-monotonic saturated incidence rate .Numerical simulations are carried out to illustrate the analytical results.

Keywords: Epidemic model, Incidence rate, Equilibria, Reproduction number, Routh-Hurwitz criterion, Lyapunov function.

MSC:34D20,49J15,92D25,92D30,93D20.

1. Introduction

Epidemic dynamics is an important method of studying the spread rules of infectious diseases qualitatively and quantitatively. It is largely based on specific properties of population growth. Epidemic models have long been an important tool for understandingand controlling the spread of infectious diseases.Authors studiedan SEIRS and an SIR epidemic model respectively with non-monotonic incidence rate and discussed the stability criteria for disease-free and endemic equilibria. Numerical simulations are also carried out[1,2]. The paper [3] discussed an epidemic model with non-monotonic incidence rate which describes the psychological effect of certain serious diseases on the community when the number of infective is getting larger. In paper[4] an epidemic model with modified non-monotonic incidence rate under a limited resource for treatment is proposed and studied. An SEIR model with nonlinear incidence rates in epidemiology is studied in [5]. O. Adebimpe et al. [6] presented an SEIRS epidemic model with nonlinear incidence rate and numerically shown the importance of the model that measures the effects of sociological, psychological or other mechanisms of the disease.

The paper is structured as follows : The SEIE epidemic model is formulated in section 2.Simplification and analytical results of the model are presented in Section 3 , 4 and 5 .Numericalresults and graphical representation are presented in Section 6. Summary andconcluding remarks round up the paper.

2. The Mathematical model

The flow of individual depicted in the following transfer diagram (figure 2.1):

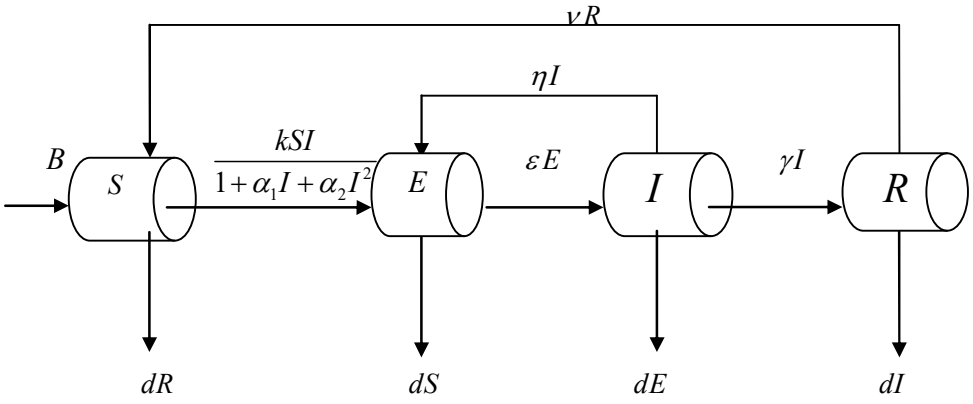

Figure 2.1: Transfer diagram for SEIE epidemic model

The symbol are used here stand for

B = recruitment rate ,

k = proportionality constant ,

ε = rate of developing infectivity ,

γ = recovery rate ,

η = effective rate ,

ν = rate of losing immunity at time t ,

d = death rate ,

α_1, α_2 = parameters which measure the effects of sociological and others mechanisms ,

where B , k , ε , γ , ν , d , α_1, α_2 are positive and $\eta \geq 0$.

3. Mathematical Analysis of the model

The differential equation corresponding to the transfer diagram are

$$\begin{aligned} \frac{dS}{dt} &= B - dS - \frac{kSI}{1+\alpha_1 I+\alpha_2 I^2} + \nu R \\ \frac{dE}{dt} &= \frac{kSI}{1+\alpha_1 I+\alpha_2 I^2} - (\varepsilon + d)E + \eta I \\ \frac{dI}{dt} &= \varepsilon E - (\gamma + d + \eta) I \\ \frac{dR}{dt} &= \gamma I - (\nu + d) R \end{aligned} \qquad (1)$$

where $N(t) = S(t) + E(t) + I(t) + R(t)$.

The feasible region D for system (1) is

$D=\{(S,E,I,R)\in \square_{+}^{4}: S\geq 0, E\geq 0, I\geq 0, R\geq 0, 0\leq S+E+I+R\leq \frac{B}{d}\}$.

This region is positively invariant with respect to system (1) .Hence , system (1) is considered mathematically and epidemiologically well posed in the region .

4. Equilibrium points

4.1 Disease-Free Equilibrium

From system (1) , we have $\frac{dS}{dt}=\frac{dE}{dt}=\frac{dI}{dt}=\frac{dR}{dt}=0$ which implies that

$$\begin{aligned} &B-dS-\frac{kSI}{1+\alpha_1 I+\alpha_2 I^2}+\nu R=0 \\ &\frac{kSI}{1+\alpha_1 I+\alpha_2 I^2}-(\varepsilon+d)E+\eta I=0 \\ &\varepsilon E-(\gamma+d+\eta)I=0 \\ &\gamma I-(\nu+d)R=0 \ . \end{aligned} \tag{2}$$

Assume that if the disease is not occur, then $I=0$ .Thus , the disease-free equilibrium E_0 is $E_0=(S_0,E_0,I_0,R_0)$, i.e. , $E_0=(\frac{B}{d},0,0,0)$.

4.2 Endemic Equilibria

If $I=I^*\neq 0$, then system (2) becomes

$$\begin{aligned} &B-dS^*-\frac{kS^*I^*}{1+\alpha_1 I^*+\alpha_2 I^{*2}}+\nu R^*=0 \\ &\frac{kS^*I^*}{1+\alpha_1 I^*+\alpha_2 I^{*2}}-(\varepsilon+d)E^*+\eta I^*=0 \\ &\varepsilon E^*-(\gamma+d+\eta)I^*=0 \\ &\gamma I^*-(\nu+d)R^*=0 \ . \end{aligned} \tag{3}$$

On solving system of equations of (3) , we obtain the endemic equilibrium $E_*=(S^*,E^*,I^*,R^*)$ such that

$$S^* = \frac{(1+\alpha_1 I^* + \alpha_2 I^{*2})[(\varepsilon+d)(\gamma+d)+\eta d]}{k\varepsilon}, \quad E^* = \frac{(\gamma + d + \eta)}{\varepsilon} I^*,$$

$$I^* = \frac{-[d\alpha_1((\varepsilon+d)(\gamma+d)+\eta d)+k((\varepsilon+d)(\gamma+d)+\eta d)-\frac{\nu k\gamma\varepsilon}{\nu+d}]+\sqrt{\Delta}}{2d\alpha_2\,[(\varepsilon+d)(\gamma+d)+\eta d]} \text{ and}$$

$$R^* = \frac{\gamma I^*}{\nu+d}$$

where $\Delta = [(d\alpha_1+k)((\varepsilon+d)(\gamma+d)+\eta d)-\frac{\nu k\gamma\varepsilon}{\nu+d}]^2 + 4d\alpha_2[(\varepsilon+d)(\gamma+d)+\eta d](R_0-1)$

and $R_0 = \frac{Bk\varepsilon}{d((\varepsilon+d)(\gamma+d)+\eta d)}$ is a basic reproduction number .

5. Stability Analysis

5.1(a) Local stability of disease-free equilibrium

Theorem 5.1.1. If $R_0 < 1$ then the disease-free equilibrium E_0 is locally asymptotically stable , E_0 is stable if $R_0 = 1$ and E_0 is unstable if $R_0 > 1$.

Proof : TheJacobian matrix at disease-free equilibrium is

$$J = \begin{pmatrix} -d & 0 & \frac{-kB}{d} & \nu \\ 0 & -(\varepsilon+d) & \frac{kB}{d}+\eta & 0 \\ 0 & \varepsilon & -(\gamma+d+\eta) & 0 \\ 0 & 0 & \gamma & -(\nu+d) \end{pmatrix}.$$

Then its characteristic equation will be

$$\begin{vmatrix} -(d+\lambda) & 0 & \frac{-kB}{d} & \nu \\ 0 & -(\varepsilon+d+\lambda) & \frac{kB}{d}+\eta & 0 \\ 0 & \varepsilon & -(\gamma+d+\eta+\lambda) & 0 \\ 0 & 0 & \gamma & -(\nu+d+\lambda) \end{vmatrix} = 0.$$

On solving ,

$$\lambda^2 + a_1 \lambda + a_2 = 0$$

where, $a_1 = (\varepsilon + \gamma + \eta + 2d) > 0$, $a_2 = ((\gamma + d)(\varepsilon + d) + \eta d - \frac{Bk\varepsilon}{d}) > 0$ if $R_0 < 1$.

Hence, by Routh-Hurwitz criterion, this theorem is proved.

5.1(b) Global stability of disease-free equilibrium

Theorem 5.1.2 . If $R_0 < 1$, the disease-free equilibrium is globally asymptotically stable otherwise unstable .

Proof: Consider Lyapunov function

$$L = \varepsilon E + (\varepsilon + d) I .$$

Then

$$\frac{dL}{dt} = ((\varepsilon + d)(\gamma + d) + \eta d)) \left[\frac{R_0}{1 + \alpha_1 I + \alpha_2 I^2} - 1 \right] I \leq 0 \quad if \quad R_0 < 1.$$

$$\frac{dL}{dt} \leq 0 .$$

Hence , the maximal compact invariant set in $\{(S, E, I, R) \in D : \frac{dL}{dt} = 0\}$ is the singleton $\{E_0\}$.

Using Lasalle's invariance principle the theorem is proved .

5.1 (c) Local stability of endemic equilibrium

Theorem 5.1.3. If $R_0 > 1$, then the endemic equilibrium is locally asymptotically stable .

Proof : Let $x = S - S^*, y = E - E^*, z = I - I^*, q = R - R^*$.

Then system (1) becomes

$$\frac{dx}{dt} = B - d(x + S^*) - \frac{k(x + S^*)(z + I^*)}{1 + \alpha_1 (z + I^*) + \alpha_2 (z + I^*)^2} + v(q + R^*)$$

$$\frac{dE}{dt} = \frac{k(x + S^*)(z + I^*)}{1 + \alpha_1 (z + I^*) + \alpha_2 (z + I^*)^2} - (\varepsilon + d)(y + E^*) + \eta (z + I^*)$$

$$\frac{dI}{dt} = \varepsilon (y + E^*) - (\gamma + d + \eta)(z + I^*)$$

$$\frac{dR}{dt} = \gamma (z + I^*) - (v + d)(q + R^*).$$

The Jacobian matrix at endemic equilibrium is

$$M = \begin{pmatrix} -\left(kI^* + d\right) & 0 & -kS^* & \nu \\ kI^* & -\left(\varepsilon + d\right) & kS^* + \eta & 0 \\ 0 & \varepsilon & -\left(\gamma + d + \eta\right) & 0 \\ 0 & 0 & \gamma & -\left(\nu + d\right) \end{pmatrix}.$$

$$\begin{vmatrix} -\left(kI^* + d + z\right) & 0 & -kS^* & \nu \\ kI^* & -\left(\varepsilon + d + z\right) & kS^* + \eta & 0 \\ 0 & \varepsilon & -\left(\gamma + d + \eta + z\right) & 0 \\ 0 & 0 & \gamma & -\left(\nu + d + z\right) \end{vmatrix} = 0.$$

On solving ,we have

$$z^4 + b_1 z^3 + b_2 z^2 + b_3 z + b_4 = 0$$

where,

$b_1 = kI^* + \varepsilon + \gamma + \nu + 4d > 0$,

$b_2 = (\nu + d)[(\gamma + \varepsilon + kI^* + 3d) + (\gamma + d)(\varepsilon + kI^* + 2d) + (kI^* + d)(\varepsilon + d) - \varepsilon kS^* > 0$,

$$b_3 = (\nu + d)((\gamma + d)(\varepsilon + d) + (\gamma + d)(kI^* + d) + (kI^* + d)(\varepsilon + d) - \varepsilon kS^*) + (kI^* + d)((\gamma + d)(\varepsilon + d) - \varepsilon k^2 S^* I^* - \varepsilon kS^*) > 0$$,

$b_4 = (\nu + d)[(kI^* + d)(\gamma + d)(\varepsilon + d) - (kI^* + d)\varepsilon kS^* - \varepsilon k^2 S^* I^*] - \nu k\varepsilon\gamma I^* > 0$

and $b_1 b_2 b_3 > {b_3}^2 + {b_1}^2 b_4$.

So, by Routh – Hurwitz criteria , all roots are negative .

Thus , the endemic equilibrium is locally asymptotically stable if $R_0 > 1$.

6. Numerical Simulation and Graphical Representation

Case I

$S(0) = 20, E(0) = 10, I(0) = 15, R(0) = 12, B = 0.01, k = 0.4, \alpha_1 = 0.7, \alpha_2 = 0, d = 0.4, \nu = 0.0033,$
$\varepsilon = 4, \gamma = 0.1, \eta = 0.9, R_0 = 0.012 < 1.$

Case II

$S(0) = 20, E(0) = 10, I(0) = 15, R(0) = 12, B = 0.01, k = 0.4, \alpha_1 = 0, \alpha_2 = 0.7, d = 0.4, \nu = 0.0033,$
$\varepsilon = 4, \gamma = 0.1, \eta = 0.9, R_0 = 0.012 < 1.$

Case III

$S(0)=20, E(0)=10, I(0)=15, R(0)=12, B=0.01, k=0.4, \alpha_1=0.7, \alpha_2=0.7, d=0.4, \nu=0.0033,$
$\varepsilon=4, \gamma=0.1, \eta=0.9, R_0=0.012<1.$

Case IV

$S(0)=20, E(0)=10, I(0)=15, R(0)=12, B=1, k=0.4, \alpha_1=0.7, \alpha_2=0, d=0.4, \nu=0.0033,$
$\varepsilon=4, \gamma=0.1, \eta=0.9, R_0=1.29>1.$

Case V

$S(0)=20, E(0)=10, I(0)=15, R(0)=12, B=1, k=0.4, \alpha_1=0, \alpha_2=0.7, d=0.4, \nu=0.0033,$
$\varepsilon=4, \gamma=0.1, \eta=0.9, R_0=1.29>1.$

Case VI

$S(0)=20, E(0)=10, I(0)=15, R(0)=12, B=3, k=0.4, \alpha_1=0.7, \alpha_2=0.7, d=0.4, \nu=0.0033,$
$\varepsilon=4, \gamma=0.1, \eta=0.9, R_0=3.87>1.$

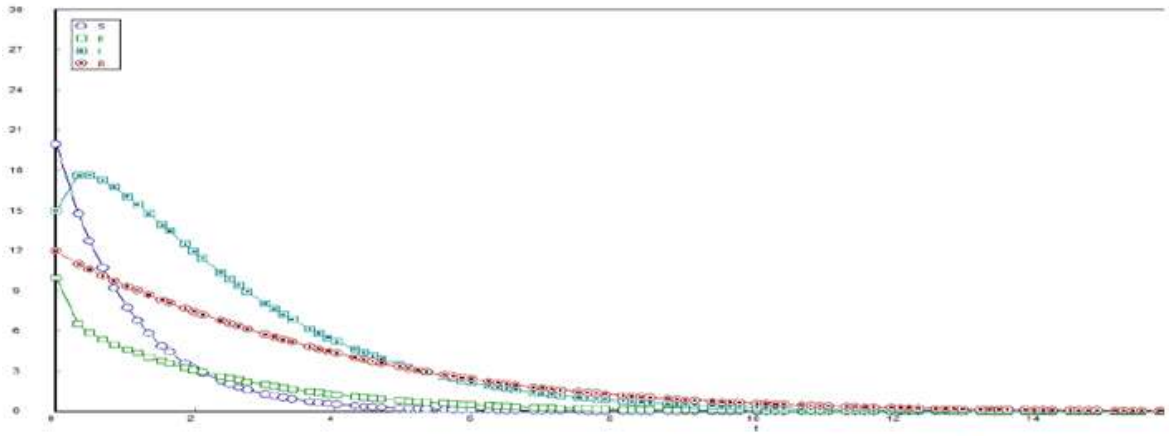

Figure 6.1. Case I : Disease dies out when $R_0<1$ at B=0.01, $\alpha_1=0.7, \alpha_2=0$

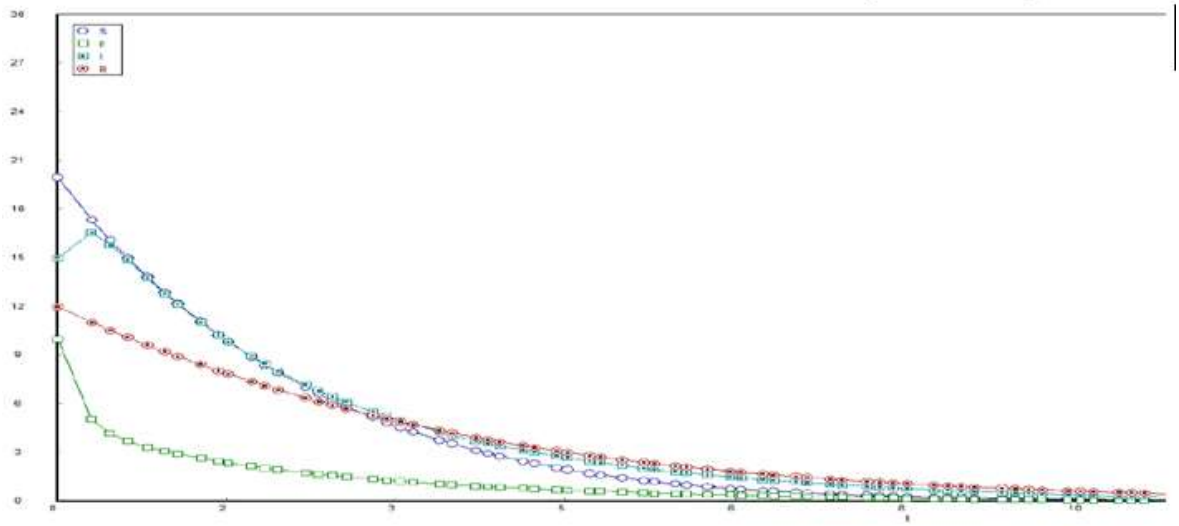

Figure 6.2. Case II : Disease dies out when $R_0<1$ at B=0.01, $\alpha_1=0, \alpha_2=0.7$

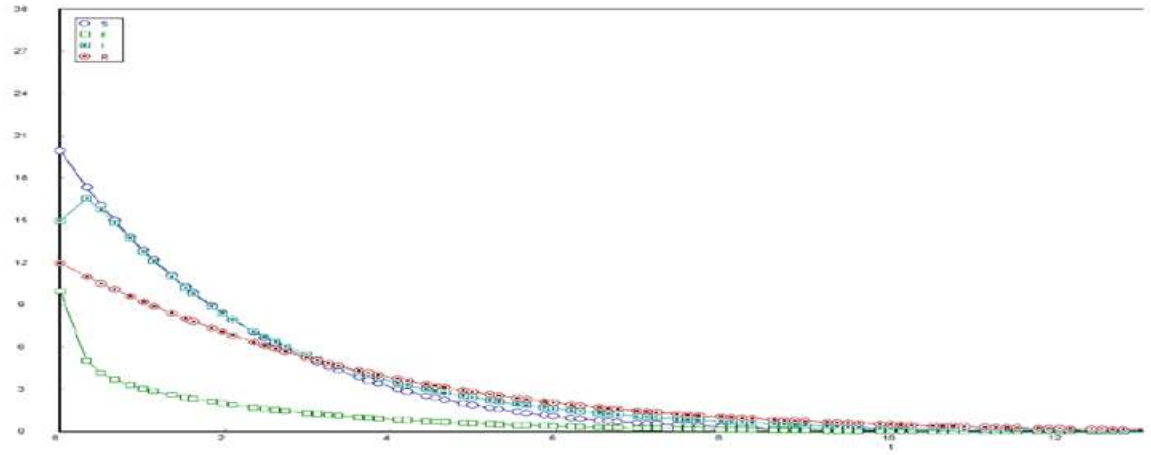

Figure 6.3. Case III : Disease dies out when $R_0 < 1$ at B=0.01, $\alpha_1 = 0.7, \alpha_2 = 0.7$

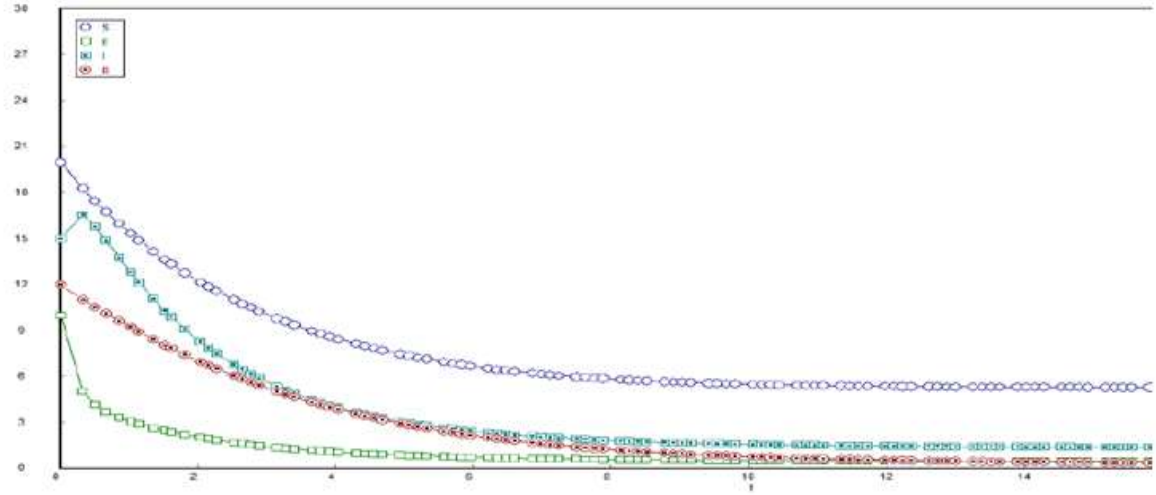

Figure 6.4. Case IV : Disease becomes endemic when $R_0 > 1$ at B=1, $\alpha_1 = 0.7, \alpha_2 = 0$

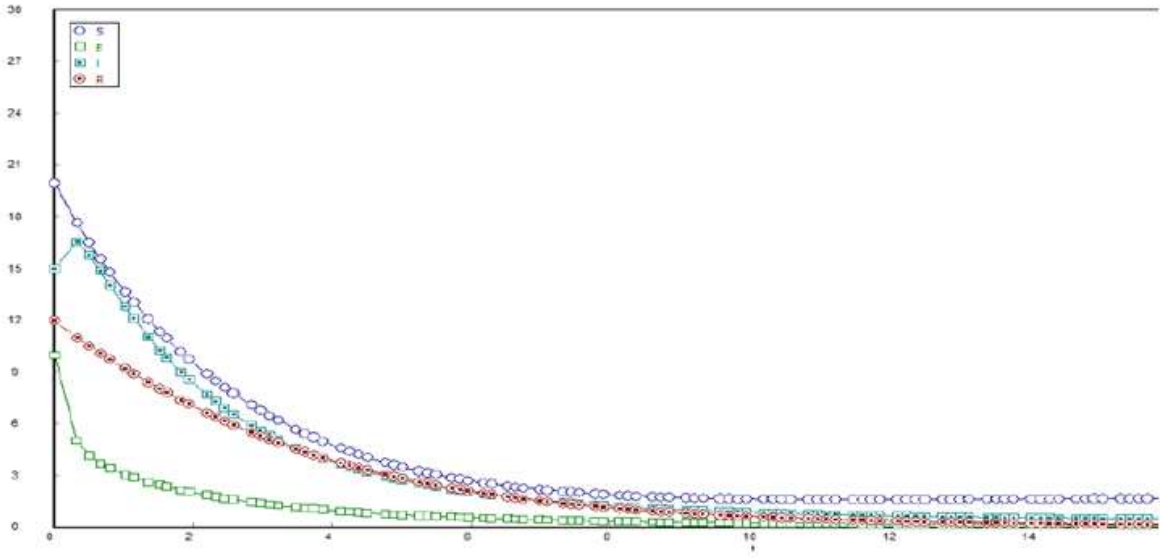

Figure 6.5. Case V : Disease becomes endemic when $R_0 > 1$ at B=1, $\alpha_1 = 0, \alpha_2 = 0.7$

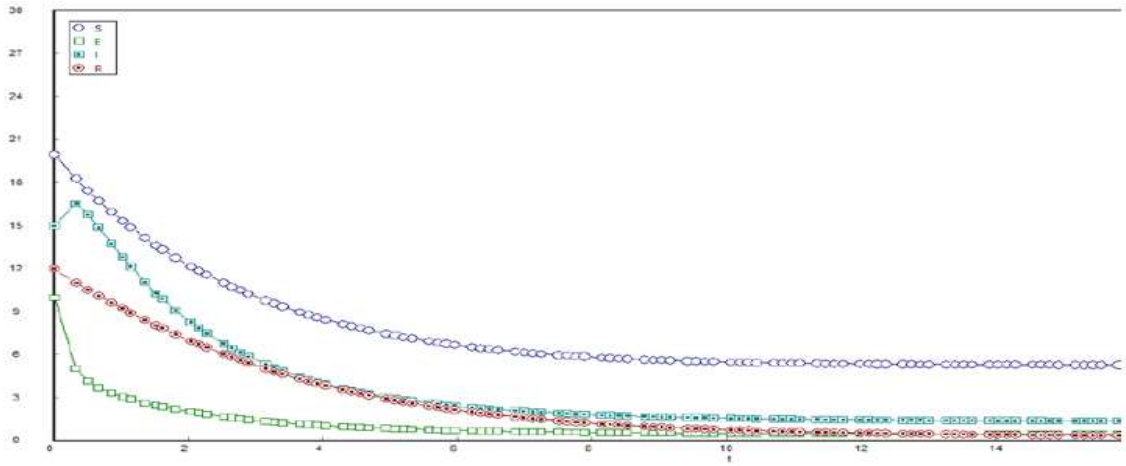

Figure 6.6. Case VI : Disease becomes endemic when $R_0 > 1$ at B=3, $\alpha_1 = 0.7, \alpha_2 = 0.7$

7. Conclusion

The non-monotone incidence rate describes the psychological or inhibitory effect: when the number of the infected individuals exceeds a certain level, the infection function decreases.In this paper, firstly I simplified an SEIE epidemic model with non-monotonic saturated incidence rate .Then I have found disease-free and endemic equilibria for the model and analyzed the stability criteria for the both equilibria . I have seen that the disease-free equilibriais locally as well as globally asymptotically stable by Routh-Hurwitz criteria and Lyapunov function respectively if $R_0 < 1$. If $R_0 > 1$ then I have discussed local asymptotically stability criteria for the endemic equilibria .Also , numerical simulations are carried out for the model with graphical representation and found that if $R_0 < 1$, the disease dies out and if $R_0 > 1$, the disease becomes endemic .The proposed model can be applied for controlling some infectious diseases in some countries or communities.

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