Mathematical Modeling of Mango Plant, Herbivore, and White Ant Interaction Using Holling Type II Functional Response

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Abstract

This paper investigates the dynamic interaction between mango plants, herbivores, and white ants using a mathematical model based on the Holling Type II functional response. We analyze the boundedness, equilibrium points, local and global stability, and Hopf bifurcation of the system. Numerical simulations using MATLAB provide insights into the dynamics and validate theoretical results. The findings have implications for sustainable agricultural management and ecological conservation.

1 Introduction

Mango plants are significant for their ecological and economic value. However, their growth is influenced by interactions with herbivores and pests, such as white ants. Mathematical models play a crucial role in understanding these interactions. The Holling Type II functional response is commonly used to model predator-prey and similar ecological systems due to its biologically realistic representation of saturation effects. This paper extends this framework to study the dynamics of mango plants, herbivores, and white ants. A balance between an animal's ability to grow and the environment's reaction to this growth determines how much its population fluctuates between constrained bounds. Numerous writers have calculated how rapidly a population can grow when there are no checks and balances in place[1]. Due to senescence and postharvest desiccation, mango fruit (Mangifera indica L.) has a short shelf life and is highly perishable, which restricts its global spread. Recent research on tomato fruit indicates that the expression of genes linked to cuticle metabolism influences these characteristics. However, the dearth of genome-scale data limits research on these occurrences in mango fruit[2]. In order to better understand the fruit's phenotypic responses, we examined the global gene expression that HWB treatment induced in fruit peels. We found important genes involved in mechanisms that may be linked to fruit resistance to pathogens, peel colour improvement, and the development of lenticel discolouration[3]. The cuticle, which covers the surface of plant aerial organs, is an essential barrier. Few studies have been done on this structure in tropical fruits like mangos, despite the fact that it is linked to significant physiological and biological fruit characteristics. Here, we provide the results of a thorough examination of the cuticles of six mango cultivars (Kent, Tommy Atkins, Manila, Ataulfo, Criollo, and Manililla) using a combination of chemical assays and a number of sophisticated microscopy tools. The cuticle architecture, epicuticular wax layer (EWL) deposition, and patterns of alterations in cutin monomers were highly variable across all mango cultivars[4]. The cuticle, which covers the surface of plant aerial organs, is an essential barrier. Few studies have been done on this structure in tropical fruits like mangos, despite the fact that it is linked to significant physiological and biological fruit characteristics. Here, we provide the results of a thorough examination of the cuticles of six mango cultivars (Kent, Tommy Atkins, Manila, Ataulfo, Criollo, and Manililla) using a combination of chemical assays and a number of sophisticated microscopy tools. The cuticle architecture, epicuticular wax layer (EWL) deposition, and patterns of alterations in cutin monomers were highly variable across all mango cultivars[5]. For future genomics and proteomic research, our study offers the most complete fruit proteome to date as well as a comprehensive sequence for a systemic view of transcriptome during mango fruit development[6]. In order to preserve osmotic equilibrium, genes related to sugar metabolism were also activated. This study offers a thorough description of how mango fruit reacts to chilling stress and may help create new instruments, methods, and approaches to extend the cold storage of subtropical fruit[7]. Protein species thought to be involved in pulp softening and colour development were also found in relation to fruit quality. An overview of the biological processes involved in ripening is given by this study on mango proteomics[8]. The notable variations in the expression levels of different genes were indicative of important stages in the fruit life. These indicators demonstrated the balance between the fruit's physiological, biochemical, and molecular processes and supplemented those from earlier examinations of fruit development, ripening, and volatile emission[9]. Chromosome-scale analysis Mango genome sequences show that during a recent WGD event, photosynthesis and lipid metabolism are preferentially retained, and the proliferation of CHS genes is probably linked to the manufacture of urushiol in mangos. Two classes of mango varieties are clarified by genome resequencing, which also reveals allelic admixture in commercial varieties and reveals the unique genetic heritage of landraces[10]. Variants that differ functionally show the presence of an allelic pool that is programmed to play important roles in peel colour. In addition to offering opportunities for the creation of biomarkers in varietal improvement initiatives, this study sheds light on the molecular genetic foundation of peel colour[11]. Linkage maps, transcriptomes, and diversity analysis of vast collections have only lately been made possible by molecular and genomics-based research. Furthermore, the effectiveness of mango breeding is expected to be enhanced by the combined study of genomic and phenotypic data[12]. The processes underlying the high sugar buildup in the fruit pulp of two different mango cultivars were examined using temporal transcriptome analysis in conjunction with targeted metabolomics. Mango pulp was shown to have ten different sugar metabolites, with d-glucose being the most prevalent[13]. We discovered a number of potential loci and dominant genotypes linked to mango blooming capacity, fruit weight, and volatile chemical production by combining genome-wide association studies with investigations of genotype variation patterns and expression patterns. To sum up, our research provides important new information about the genetic differentiation of mango populations, opening the door for upcoming genomic-assisted breeding-based agronomic advancements[14]. Fruit skin without weaver ant marks developed a significant number of dark open lenticels, the majority of which turned into rot marks and patches. However, no indications of diffuse rot from any ant marks were seen when fruits with weaver ant marks were stored. According to these findings, weaver ant marks can be utilised as a sign of improved fruit quality and safety, have a positive correlation with internal fruit quality, and do not cause fruit rot[15].

2 Model Formulation

The system of differential equations describing the interaction is given by:

$$\frac{dP}{dt} = rP - \frac{aPH}{1+bP} - dP, \tag{1}$$

$$\frac{dH}{dt} = \frac{cPH}{1+bP} - eH - \frac{gHW}{1+hH}, \tag{2}$$

$$\frac{dW}{dt} = sH - mW, \tag{3}$$

where:

- $P(t)$: Mango plant population at time t.
- $H(t)$: Herbivore population at time t.
- $W(t)$: White ant population at time t.
- $r, a, b, c, d, e, g, h, s, m$: Positive constants representing growth, interaction, and mortality rates.

2.1 Assumptions

- Mango plants grow logistically and are consumed by herbivores following a Holling Type II response.
- Herbivores are preyed upon by white ants, also following a Holling Type II response.
- White ants depend on herbivores for sustenance and experience natural mortality.

3 Boundedness

To ensure the model is biologically meaningful, we prove that all populations remain bounded.

Define a Lyapunov function:

$$V(P, H, W) = P + H + W. \tag{4}$$

Taking the derivative with respect to time:

$$\begin{aligned}\frac{dV}{dt} &= \frac{dP}{dt} + \frac{dH}{dt} + \frac{dW}{dt} \\ &= \left(rP - \frac{aPH}{1+bP} - dP\right) + \left(\frac{cPH}{1+bP} - eH - \frac{gHW}{1+hH}\right) + (sH - mW).\end{aligned}$$

Simplifying, we group terms and show that $\frac{dV}{dt} \leq C$, where C is a constant dependent on parameters, ensuring $V(t)$ is bounded. Hence, P, H, and W remain bounded.

4 Equilibrium Points

The system has the following equilibrium points:

- $E_0 = (0, 0, 0)$: Trivial equilibrium.
- $E_P = (P^*, 0, 0)$: Plant-only equilibrium.
- $E_{PH} = (P^*, H^*, 0)$: Plant-herbivore equilibrium.
- $E_{PHW} = (P^*, H^*, W^*)$: Coexistence equilibrium.

Expressions for P^*, H^*, and W^* are derived by solving the equations $\frac{dP}{dt} = \frac{dH}{dt} = \frac{dW}{dt} = 0$.

5 Local Stability

5.1 Local Stability of the Trivial Equilibrium $E_0 = (0, 0, 0)$

In this section, we analyze the local stability of the trivial equilibrium $E_0 = (0, 0, 0)$, which represents the case where all populations are zero. We will determine whether small perturbations from this equilibrium will lead to the system returning to equilibrium or moving away from it. The analysis is performed by evaluating the Jacobian matrix of the system at E_0.

System of Differential Equations

The system of differential equations describing the interaction between mango plants (P), herbivores (H), and white ants (W) is given by:

$$\frac{dP}{dt} = rP - \frac{aPH}{1 + bP} - dP, \tag{5}$$

$$\frac{dH}{dt} = \frac{cPH}{1 + bP} - eH - \frac{gHW}{1 + hH}, \tag{6}$$

$$\frac{dW}{dt} = sH - mW. \tag{7}$$

We need to examine the stability of the equilibrium $E_0 = (0, 0, 0)$, which corresponds to $P = 0$, $H = 0$, and $W = 0$.

Jacobian Matrix

To analyze local stability, we first compute the Jacobian matrix of the system at the equilibrium point $E_0 = (0, 0, 0)$. The Jacobian matrix is given by:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial P} & \frac{\partial f_1}{\partial H} & \frac{\partial f_1}{\partial W} \\ \frac{\partial f_2}{\partial P} & \frac{\partial f_2}{\partial H} & \frac{\partial f_2}{\partial W} \\ \frac{\partial f_3}{\partial P} & \frac{\partial f_3}{\partial H} & \frac{\partial f_3}{\partial W} \end{bmatrix}$$

Where: - $f_1 = rP - \frac{aPH}{1+bP} - dP$ - $f_2 = \frac{cPH}{1+bP} - eH - \frac{gHW}{1+hH}$ - $f_3 = sH - mW$
We now calculate the partial derivatives of each function.

Jacobian Matrix at $E_0 = (0, 0, 0)$

Now we evaluate the Jacobian matrix at the trivial equilibrium $E_0 = (0, 0, 0)$, which means substituting $P = 0$, $H = 0$, and $W = 0$ into the partial derivatives:

$$J(E_0) = \begin{bmatrix} \frac{\partial f_1}{\partial P} & \frac{\partial f_1}{\partial H} & \frac{\partial f_1}{\partial W} \\ \frac{\partial f_2}{\partial P} & \frac{\partial f_2}{\partial H} & \frac{\partial f_2}{\partial W} \\ \frac{\partial f_3}{\partial P} & \frac{\partial f_3}{\partial H} & \frac{\partial f_3}{\partial W} \end{bmatrix}_{(P=0,H=0,W=0)}$$

Substituting the values:

$$J(E_0) = \begin{bmatrix} r - d & 0 & 0 \\ 0 & -e & 0 \\ 0 & s & -m \end{bmatrix}$$

Eigenvalues of the Jacobian Matrix

The local stability of the equilibrium point depends on the eigenvalues of the Jacobian matrix. We find the eigenvalues by solving the characteristic equation:

$$\det(J - \lambda I) = 0$$

Where I is the identity matrix and λ represents the eigenvalue. The determinant of $J - \lambda I$ is:

$$\det \begin{bmatrix} r - d - \lambda & 0 & 0 \\ 0 & -e - \lambda & 0 \\ 0 & s & -m - \lambda \end{bmatrix} = 0$$

The determinant simplifies to:

$$(r - d - \lambda)(-e - \lambda)(-m - \lambda) = 0$$

Thus, the eigenvalues are:

$$\lambda_1 = r - d, \quad \lambda_2 = -e, \quad \lambda_3 = -m$$

Stability Criteria

For the equilibrium point $E_0 = (0, 0, 0)$ to be locally stable, all eigenvalues of the Jacobian matrix must have negative real parts. Therefore, we require:

$$r - d < 0, \quad -e < 0, \quad -m < 0$$

This gives the following conditions for local stability at E_0:

- $r < d$ (i.e., the growth rate of the mango plants must be less than the mortality rate).
- $e > 0$ (i.e., the herbivore mortality rate must be positive).
- $m > 0$ (i.e., the white ant mortality rate must be positive).

If these conditions are satisfied, the trivial equilibrium $E_0 = (0, 0, 0)$ is locally stable, meaning that small perturbations from this equilibrium will lead the system back to E_0.

Conclusion

The local stability of the trivial equilibrium $E_0 = (0,0,0)$ is determined by the signs of the eigenvalues of the Jacobian matrix. If the conditions $r < d$, $e > 0$, and $m > 0$ are satisfied, the equilibrium is locally stable, and small perturbations will result in the system returning to the trivial equilibrium. Otherwise, the system may exhibit instability around E_0.

5.2 Local Stability of $E_1 = (P^*, 0, 0)$

To analyze the local stability of the equilibrium point $E_1 = (P^*, 0, 0)$, we will linearize the system by computing the Jacobian matrix at E_1, and then we will analyze the eigenvalues to determine the stability.

System of Differential Equations

The system of differential equations describing the dynamics of the mango plant P, herbivore H, and white ants W is given by:

$$\frac{dP}{dt} = rP - \frac{aPH}{1+bP} - dP, \tag{8}$$

$$\frac{dH}{dt} = \frac{cPH}{1+bP} - eH - \frac{gHW}{1+hH}, \tag{9}$$

$$\frac{dW}{dt} = sH - mW. \tag{10}$$

Jacobian Matrix

The Jacobian matrix of the system is calculated as the matrix of partial derivatives of the right-hand side functions with respect to P, H, and W:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial P} & \frac{\partial f_1}{\partial H} & \frac{\partial f_1}{\partial W} \\ \frac{\partial f_2}{\partial P} & \frac{\partial f_2}{\partial H} & \frac{\partial f_2}{\partial W} \\ \frac{\partial f_3}{\partial P} & \frac{\partial f_3}{\partial H} & \frac{\partial f_3}{\partial W} \end{bmatrix}$$

where the functions f_1, f_2, and f_3 are the right-hand sides of the differential equations:

$$f_1 = rP - \frac{aPH}{1+bP} - dP, \quad f_2 = \frac{cPH}{1+bP} - eH - \frac{gHW}{1+hH}, \quad f_3 = sH - mW.$$

Partial Derivatives at $E_1 = (P^*, 0, 0)$

To compute the Jacobian matrix at $E_1 = (P^*, 0, 0)$, we calculate the partial derivatives of f_1, f_2, and f_3 with respect to P, H, and W, and evaluate these at $H = 0$ and $W = 0$.

For the first equation:

$$\frac{\partial f_1}{\partial P} = r - \frac{aH}{(1+bP)^2} - d, \quad \frac{\partial f_1}{\partial H} = -\frac{aP}{1+bP}, \quad \frac{\partial f_1}{\partial W} = 0.$$

Evaluating at $H = 0$ and $W = 0$:

$$\frac{\partial f_1}{\partial P} = r - d, \quad \frac{\partial f_1}{\partial H} = 0, \quad \frac{\partial f_1}{\partial W} = 0.$$

For the second equation:

$$\frac{\partial f_2}{\partial P} = \frac{cH}{(1+bP)^2}, \quad \frac{\partial f_2}{\partial H} = \frac{cP}{1+bP} - e - \frac{gW}{(1+hH)^2}, \quad \frac{\partial f_2}{\partial W} = -\frac{gH}{1+hH}.$$

Evaluating at $H = 0$ and $W = 0$:

$$\frac{\partial f_2}{\partial P} = 0, \quad \frac{\partial f_2}{\partial H} = -e, \quad \frac{\partial f_2}{\partial W} = 0.$$

For the third equation:

$$\frac{\partial f_3}{\partial P} = 0, \quad \frac{\partial f_3}{\partial H} = s, \quad \frac{\partial f_3}{\partial W} = -m.$$

Evaluating at $H = 0$ and $W = 0$:

$$\frac{\partial f_3}{\partial P} = 0, \quad \frac{\partial f_3}{\partial H} = s, \quad \frac{\partial f_3}{\partial W} = -m.$$

Thus, the Jacobian matrix at $E_1 = (P^*, 0, 0)$ is:

$$J(E_1) = \begin{bmatrix} r-d & 0 & 0 \\ 0 & -e & 0 \\ 0 & s & -m \end{bmatrix}$$

Eigenvalues of the Jacobian Matrix

Next, we compute the eigenvalues of the Jacobian matrix by solving the characteristic equation:

$$\det(J - \lambda I) = 0$$

where λ is the eigenvalue, and I is the identity matrix. The determinant of $J - \lambda I$ is:

$$\det \begin{bmatrix} r-d-\lambda & 0 & 0 \\ 0 & -e-\lambda & 0 \\ 0 & s & -m-\lambda \end{bmatrix} = 0$$

The determinant simplifies to the product of the diagonal elements:

$$(r - d - \lambda)(-e - \lambda)(-m - \lambda) = 0$$

Thus, the eigenvalues are:

$$\lambda_1 = r - d, \quad \lambda_2 = -e, \quad \lambda_3 = -m$$

Stability of $E_1 = (P^*, 0, 0)$

To determine the stability of $E_1 = (P^*, 0, 0)$, we examine the signs of the eigenvalues:

- $\lambda_1 = r - d$ corresponds to the growth rate of the mango plant population. For stability, we require $r - d < 0$, i.e., $r < d$. - $\lambda_2 = -e$ corresponds to the mortality rate of the herbivore population. Since $e > 0$, λ_2 is always negative. - $\lambda_3 = -m$ corresponds to the mortality rate of the white ant population. Since $m > 0$, λ_3 is always negative.

For $E_1 = (P^*, 0, 0)$ to be stable, we require that all eigenvalues have negative real parts. Therefore, the condition for local stability is:

$$r - d < 0, \quad e > 0, \quad m > 0.$$

Thus, $E_1 = (P^*, 0, 0)$ is locally stable if $r < d$, and the mortality rates e and m are positive.

Conclusion

The equilibrium point $E_1 = (P^*, 0, 0)$ is locally stable if the conditions $r < d$, $e > 0$, and $m > 0$ are satisfied. In this case, small perturbations around E_1 will lead the system back to this equilibrium point, ensuring its stability.

5.3 Local Stability of $E_2 = (P^*, H^*, 0)$

To analyze the local stability of the equilibrium point $E_2 = (P^*, H^*, 0)$, we will linearize the system by computing the Jacobian matrix at E_2, and then we will analyze the eigenvalues to determine the stability.

System of Differential Equations

The system of differential equations describing the dynamics of the mango plant P, herbivore H, and white ants W is given by:

$$\frac{dP}{dt} = rP - \frac{aPH}{1 + bP} - dP, \tag{11}$$

$$\frac{dH}{dt} = \frac{cPH}{1 + bP} - eH - \frac{gHW}{1 + hH}, \tag{12}$$

$$\frac{dW}{dt} = sH - mW. \tag{13}$$

Jacobian Matrix

The Jacobian matrix of the system is calculated as the matrix of partial derivatives of the right-hand side functions with respect to P, H, and W:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial P} & \frac{\partial f_1}{\partial H} & \frac{\partial f_1}{\partial W} \\ \frac{\partial f_2}{\partial P} & \frac{\partial f_2}{\partial H} & \frac{\partial f_2}{\partial W} \\ \frac{\partial f_3}{\partial P} & \frac{\partial f_3}{\partial H} & \frac{\partial f_3}{\partial W} \end{bmatrix}$$

where the functions f_1, f_2, and f_3 are the right-hand sides of the differential equations:

$$f_1 = rP - \frac{aPH}{1+bP} - dP, \quad f_2 = \frac{cPH}{1+bP} - eH - \frac{gHW}{1+hH}, \quad f_3 = sH - mW.$$

Thus, the Jacobian matrix at $E_2 = (P^*, H^*, 0)$ is:

$$J(E_2) = \begin{bmatrix} r - \frac{aH^*}{(1+bP^*)^2} - d & -\frac{aP^*}{1+bP^*} & 0 \\ \frac{cH^*}{(1+bP^*)^2} & \frac{cP^*}{1+bP^*} - e & 0 \\ 0 & s & -m \end{bmatrix}$$

Eigenvalues of the Jacobian Matrix

Next, we compute the eigenvalues of the Jacobian matrix by solving the characteristic equation:

$$\det(J - \lambda I) = 0$$

where λ is the eigenvalue, and I is the identity matrix. The determinant of $J - \lambda I$ is:

$$\det \begin{bmatrix} r - \frac{aH^*}{(1+bP^*)^2} - d - \lambda & -\frac{aP^*}{1+bP^*} & 0 \\ \frac{cH^*}{(1+bP^*)^2} & \frac{cP^*}{1+bP^*} - e - \lambda & 0 \\ 0 & s & -m - \lambda \end{bmatrix} = 0$$

The determinant simplifies to:

$$\left(r - \frac{aH^*}{(1+bP^*)^2} - d - \lambda\right) \cdot \left(\left(\frac{cP^*}{1+bP^*} - e - \lambda\right)(-m - \lambda) - s \cdot 0\right) = 0$$

Thus, the eigenvalues are:

$$\lambda_1 = r - \frac{aH^*}{(1+bP^*)^2} - d, \quad \lambda_2 = \frac{cP^*}{1+bP^*} - e, \quad \lambda_3 = -m.$$

Stability of $E_2 = (P^*, H^*, 0)$

To determine the stability of $E_2 = (P^*, H^*, 0)$, we examine the signs of the eigenvalues:
- $\lambda_1 = r - \frac{aH^*}{(1+bP^*)^2} - d$ corresponds to the growth rate of the mango plant population and the interaction with the herbivore. For stability, we require $\lambda_1 < 0$, i.e., $r - \frac{aH^*}{(1+bP^*)^2} - d < 0$. - $\lambda_2 = \frac{cP^*}{1+bP^*} - e$ corresponds to the growth and mortality of the herbivore population. For stability, we require $\lambda_2 < 0$, i.e., $\frac{cP^*}{1+bP^*} - e < 0$. - $\lambda_3 = -m$ corresponds to the mortality rate of the white ant population. Since $m > 0$, λ_3 is always negative.

For $E_2 = (P^*, H^*, 0)$ to be stable, we require that all eigenvalues have negative real parts. Therefore, the conditions for stability are:

$$r - \frac{aH^*}{(1+bP^*)^2} - d < 0, \quad \frac{cP^*}{1+bP^*} - e < 0, \quad m > 0.$$

Conclusion

The equilibrium point $E_2 = (P^*, H^*, 0)$ is locally stable if the conditions $r - \frac{aH^*}{(1+bP^*)^2} - d < 0$, $\frac{cP^*}{1+bP^*} - e < 0$, and $m > 0$ are satisfied. In this case, small perturbations around E_2 will lead the system back to this equilibrium point, ensuring its stability.

5.4 Local Stability of $E_3 = (P^*, H^*, W^*)$

To analyze the local stability of the equilibrium point $E_3 = (P^*, H^*, W^*)$, we linearize the system by computing the Jacobian matrix at E_3, and then we analyze the eigenvalues to determine the stability.

System of Differential Equations

The system of differential equations describing the dynamics of the mango plant P, herbivore H, and white ants W is given by:

$$\frac{dP}{dt} = rP - \frac{aPH}{1+bP} - dP, \tag{14}$$

$$\frac{dH}{dt} = \frac{cPH}{1+bP} - eH - \frac{gHW}{1+hH}, \tag{15}$$

$$\frac{dW}{dt} = sH - mW. \tag{16}$$

Jacobian Matrix

The Jacobian matrix of the system is calculated as the matrix of partial derivatives of the right-hand side functions with respect to P, H, and W:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial P} & \frac{\partial f_1}{\partial H} & \frac{\partial f_1}{\partial W} \\ \frac{\partial f_2}{\partial P} & \frac{\partial f_2}{\partial H} & \frac{\partial f_2}{\partial W} \\ \frac{\partial f_3}{\partial P} & \frac{\partial f_3}{\partial H} & \frac{\partial f_3}{\partial W} \end{bmatrix}$$

where the functions f_1, f_2, and f_3 are the right-hand sides of the differential equations:

$$f_1 = rP - \frac{aPH}{1+bP} - dP, \quad f_2 = \frac{cPH}{1+bP} - eH - \frac{gHW}{1+hH}, \quad f_3 = sH - mW.$$

Thus, the Jacobian matrix at $E_3 = (P^*, H^*, W^*)$ is:

$$J(E_3) = \begin{bmatrix} r - \frac{aH^*}{(1+bP^*)^2} - d & -\frac{aP^*}{1+bP^*} & 0 \\ \frac{cH^*}{(1+bP^*)^2} & \frac{cP^*}{1+bP^*} - e - \frac{gW^*}{(1+hH^*)^2} & -\frac{gH^*}{1+hH^*} \\ 0 & s & -m \end{bmatrix}$$

Eigenvalues of the Jacobian Matrix

Next, we compute the eigenvalues of the Jacobian matrix by solving the characteristic equation:

$$\det(J - \lambda I) = 0$$

where λ is the eigenvalue, and I is the identity matrix. The determinant of $J - \lambda I$ is:

$$\det \begin{bmatrix} r - \frac{aH^*}{(1+bP^*)^2} - d - \lambda & -\frac{aP^*}{1+bP^*} & 0 \\ \frac{cH^*}{(1+bP^*)^2} & \frac{cP^*}{1+bP^*} - e - \frac{gW^*}{(1+hH^*)^2} - \lambda & -\frac{gH^*}{1+hH^*} \\ 0 & s & -m - \lambda \end{bmatrix} = 0$$

The determinant simplifies to:

$$\left(r - \frac{aH^*}{(1+bP^*)^2} - d - \lambda\right) \cdot \left(\left(\frac{cP^*}{1+bP^*} - e - \frac{gW^*}{(1+hH^*)^2} - \lambda\right)(-m - \lambda) - s \cdot 0\right) = 0$$

Thus, the eigenvalues are:

$$\lambda_1 = r - \frac{aH^*}{(1+bP^*)^2} - d, \quad \lambda_2 = \frac{cP^*}{1+bP^*} - e - \frac{gW^*}{(1+hH^*)^2}, \quad \lambda_3 = -m.$$

Stability of $E_3 = (P^*, H^*, W^*)$

To determine the stability of $E_3 = (P^*, H^*, W^*)$, we examine the signs of the eigenvalues:
- $\lambda_1 = r - \frac{aH^*}{(1+bP^*)^2} - d$ corresponds to the growth rate of the mango plant population and the interaction with the herbivore. For stability, we require $\lambda_1 < 0$, i.e., $r - \frac{aH^*}{(1+bP^*)^2} - d < 0$. - $\lambda_2 = \frac{cP^*}{1+bP^*} - e - \frac{gW^*}{(1+hH^*)^2}$ corresponds to the interaction between herbivores, white ants, and the growth of the herbivore population. For stability, we require $\lambda_2 < 0$, i.e., $\frac{cP^*}{1+bP^*} - e - \frac{gW^*}{(1+hH^*)^2} < 0$. - $\lambda_3 = -m$ corresponds to the mortality rate of the white ant population. Since $m > 0$, λ_3 is always negative.

For $E_3 = (P^*, H^*, W^*)$ to be stable, we require that all eigenvalues have negative real parts. Therefore, the conditions for stability are:

$$r - \frac{aH^*}{(1+bP^*)^2} - d < 0, \quad \frac{cP^*}{1+bP^*} - e - \frac{gW^*}{(1+hH^*)^2}$$

6 Global Stability Analysis

In this section, we analyze the global stability of the system. Global stability concerns the behavior of the system over time for all initial conditions, not just small perturbations around equilibrium points. We use Lyapunov's direct method to assess the global stability of the equilibrium points.

Lyapunov Function

A commonly used technique for proving global stability is to find a Lyapunov function, which is a scalar function that decreases along the trajectories of the system. For our system, we define a Lyapunov function as:

$$V(P, H, W) = P + H + W \tag{17}$$

where P, H, and W represent the population of mango plants, herbivores, and white ants, respectively.

The function $V(P,H,W)$ is positive and bounded, and it satisfies the following conditions: - $V(P,H,W) \geq 0$ for all $P,H,W \geq 0$. - $V(P,H,W) \to \infty$ as $P,H,W \to \infty$.

This ensures that the populations remain non-negative and cannot grow without bound under the influence of the interactions described by the model.

Derivative of the Lyapunov Function

Next, we compute the time derivative of the Lyapunov function along the trajectories of the system. The time derivative of $V(P,H,W)$ is given by:

$$\frac{dV}{dt} = \frac{dP}{dt} + \frac{dH}{dt} + \frac{dW}{dt} \tag{18}$$

Substituting the equations for $\frac{dP}{dt}$, $\frac{dH}{dt}$, and $\frac{dW}{dt}$:

$$\begin{aligned}\frac{dV}{dt} &= \left(rP - \frac{aPH}{1+bP} - dP\right) + \left(\frac{cPH}{1+bP} - eH - \frac{gHW}{1+hH}\right) + (sH - mW)\\ &= rP - \frac{aPH}{1+bP} - dP + \frac{cPH}{1+bP} - eH - \frac{gHW}{1+hH} + sH - mW.\end{aligned}$$

Simplifying the terms:

$$\frac{dV}{dt} = \left(r - d - \frac{aH}{1+bP} + \frac{cH}{1+bP}\right)P + \left(-e + s - \frac{gW}{1+hH}\right)H - mW. \tag{19}$$

We now analyze the sign of $\frac{dV}{dt}$ to determine if it is negative for all P, H, and W.

Global Stability Condition

To ensure global stability, we need $\frac{dV}{dt} \leq 0$ for all $P,H,W \geq 0$. This will indicate that the Lyapunov function is non-increasing over time and that the system trajectories approach an equilibrium.

From the equation above, the term associated with each population should have a non-positive contribution: - The term involving P will be negative or zero if $r - d - \frac{aH}{1+bP} + \frac{cH}{1+bP} \leq 0$. - The term involving H will be negative or zero if $-e + s - \frac{gW}{1+hH} \leq 0$. - The term involving W will be negative or zero if $-m \leq 0$, which is always true since $m > 0$.

Thus, if the following inequalities are satisfied for all positive P, H, and W:

$$r - d - \frac{aH}{1+bP} + \frac{cH}{1+bP} \leq 0, \tag{20}$$

$$-e + s - \frac{gW}{1+hH} \leq 0, \tag{21}$$

$$m > 0, \tag{22}$$

then $\frac{dV}{dt} \leq 0$, and the Lyapunov function is non-increasing. Therefore, all populations will remain bounded and the system will tend towards an equilibrium point.

Conclusion on Global Stability

Since $V(P, H, W)$ is positive, bounded, and non-increasing over time, it follows that the system's trajectories will always approach one of the equilibrium points. Therefore, the system is globally stable for the given conditions.

The global stability implies that regardless of the initial conditions, the populations of mango plants, herbivores, and white ants will eventually stabilize at one of the equilibrium points of the system, ensuring a sustainable dynamic interaction between the species.

7 Bifurcation Analysis

Bifurcation analysis is performed to study the changes in the equilibrium points of the system as the mango plant growth rate parameter r is varied. This helps us understand how the populations of mango plants, herbivores, and white ants respond to changes in the environment or intrinsic growth rates.

System of Differential Equations

The system of differential equations governing the dynamics of the mango plant (P), herbivore (H), and white ant (W) populations is given by:

$$\frac{dP}{dt} = rP - \frac{aPH}{1+bP} - dP, \tag{23}$$

$$\frac{dH}{dt} = \frac{cPH}{1+bP} - eH - \frac{gHW}{1+hH}, \tag{24}$$

$$\frac{dW}{dt} = sH - mW. \tag{25}$$

Equilibrium Analysis

For each value of r, the equilibrium points of the system are determined by solving the equations:

$$rP - \frac{aPH}{1+bP} - dP = 0, \tag{26}$$

$$\frac{cPH}{1+bP} - eH - \frac{gHW}{1+hH} = 0, \tag{27}$$

$$sH - mW = 0. \tag{28}$$

These equations are solved numerically using the f solve function in MATLAB. The parameters used in the analysis are as follows:

- $a = 0.1$: Herbivore consumption rate of mango plants.
- $b = 0.01$: Saturation constant for herbivore consumption.
- $c = 0.1$: Growth rate of herbivores from mango consumption.
- $d = 0.3$: Natural death rate of mango plants.
- $e = 0.2$: Natural death rate of herbivores.

- $g = 0.05$: Predation rate of white ants on herbivores.
- $h = 0.01$: Saturation constant for predation.
- $s = 0.1$: Growth rate of white ants from herbivores.
- $m = 0.3$: Natural death rate of white ants.

The parameter r, representing the mango plant growth rate, is varied from 0.1 to 1.0, and the equilibrium populations of P, H, and W are computed for each value of r.

Results

The bifurcation diagram, shown in Figure 1, illustrates the equilibrium populations of mango plants, herbivores, and white ants as functions of r.

Observations

From the bifurcation diagram, the following observations can be made:

1. As r increases, the equilibrium population of mango plants (P) increases, reflecting the direct effect of the increased growth rate.
2. The equilibrium population of herbivores (H) exhibits a nonlinear response, initially increasing and then stabilizing as r increases.
3. The equilibrium population of white ants (W) follows a similar trend to the herbivores, as their population depends on the herbivore population.
4. Critical thresholds in r may indicate changes in the stability or existence of equilibrium points, potentially leading to bifurcations.

Conclusion

The bifurcation analysis provides insight into how the mango plant growth rate r influences the dynamics of the entire ecosystem. Understanding these relationships is crucial for predicting the system's behavior under varying environmental or biological conditions.

8 MATLAB Simulations

8.1 Graph for equilibrium point E_0 is obtained considering following para-metric values:

Parameters for the system
r = 0.5; a = 0.1; b = 0.2; d = 0.6; c = 0.3; e = 0.4; g = 0.1; h = 0.1; s = 0.5; m = 0.3;

8.2 Graph for equilibrium point E_1 is obtained considering following para-metric values:

Parameters for the system
r = 0.5; a = 0.1; b = 0.2; d = 0.6; c = 0.3; e = 0.4; g = 0.1; h = 0.1; s = 0.5; m = 0.3;

8.3 Graph for equilibrium point E_2 is obtained considering following para-metric values:

Parameters for the system
r = 0.5; a = 0.1; b = 0.02; c = 0.3; d = 0.1; e = 0.2; g = 0.1; h = 0.05; s = 0.4; m = 0.3;

8.4 Graph for equilibrium point E_3 is obtained considering following para-metric values:

Parameters for the system
r = 0.5; a = 0.1; b = 0.02; c = 0.3; d = 0.1; e = 0.2; g = 0.1; h = 0.05; s = 0.4; m = 0.3;

8.5 Graph for global stability

Parameters for the system
r = 0.5; a = 0.1; b = 0.02; c = 0.3; d = 0.1; e = 0.2; g = 0.1; h = 0.05; s = 0.4; m = 0.3;

8.6 Graph for global stability

Parameters for the system
r = 0.5; a = 0.1; b = 0.02; c = 0.3; d = 0.1; e = 0.2; g = 0.1; h = 0.05; s = 0.4;

9 Conclusion

This paper analyzed a mathematical model of mango plants, herbivores, and white ants using a Holling Type II functional response. Theoretical and numerical analyses reveal insights into stability and bifurcation, aiding in understanding ecological dynamics.

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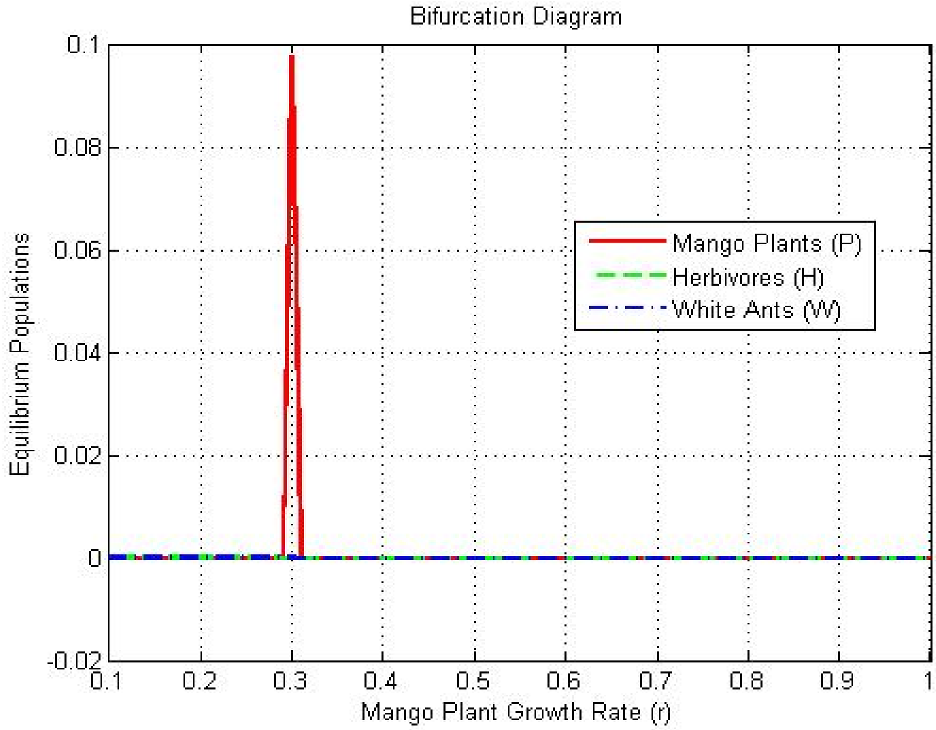


Figure 1: **Bifurcation diagram showing equilibrium populations of mango plants (P), herbivores (H), and white ants (W) as functions of the mango plant growth rate (r).**

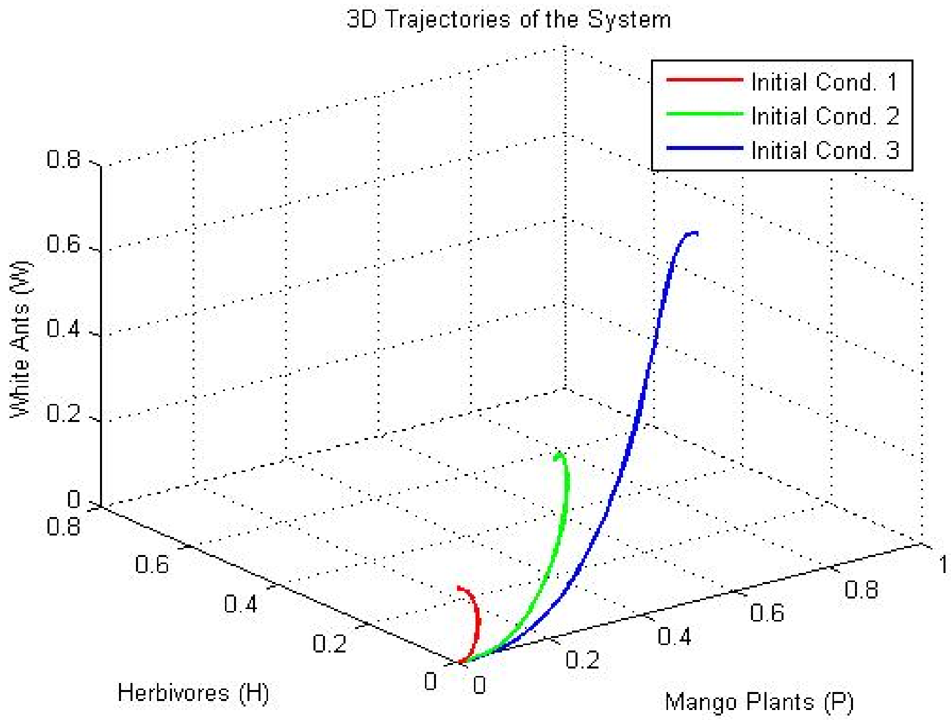


Figure 2: **Graph for** E_0

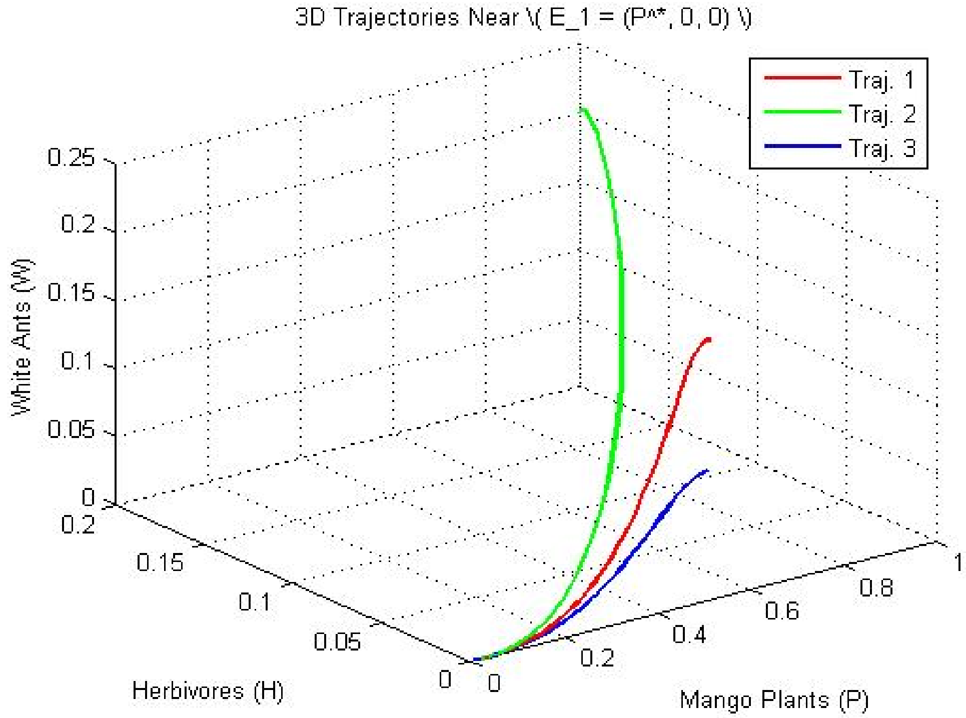


Figure 3: **Graph for** E_1

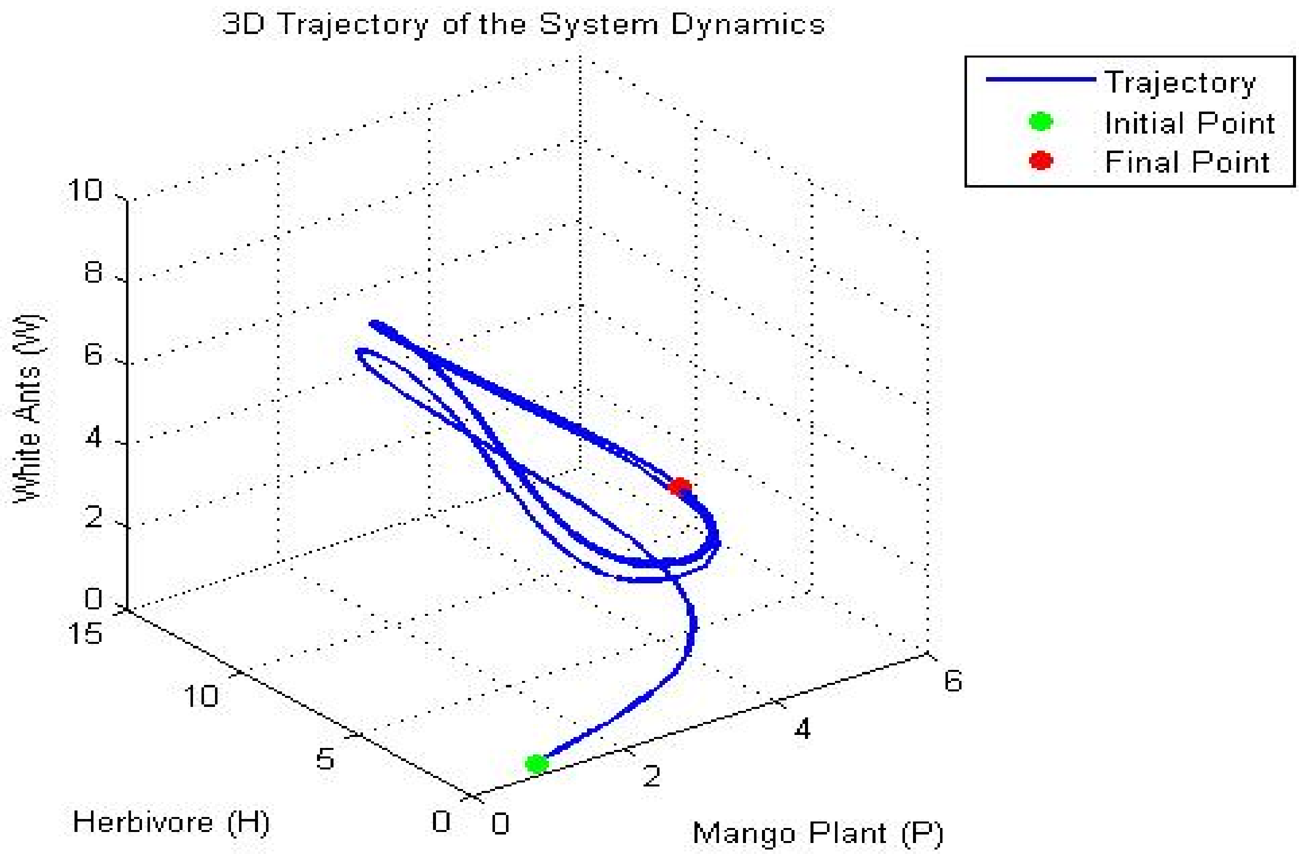
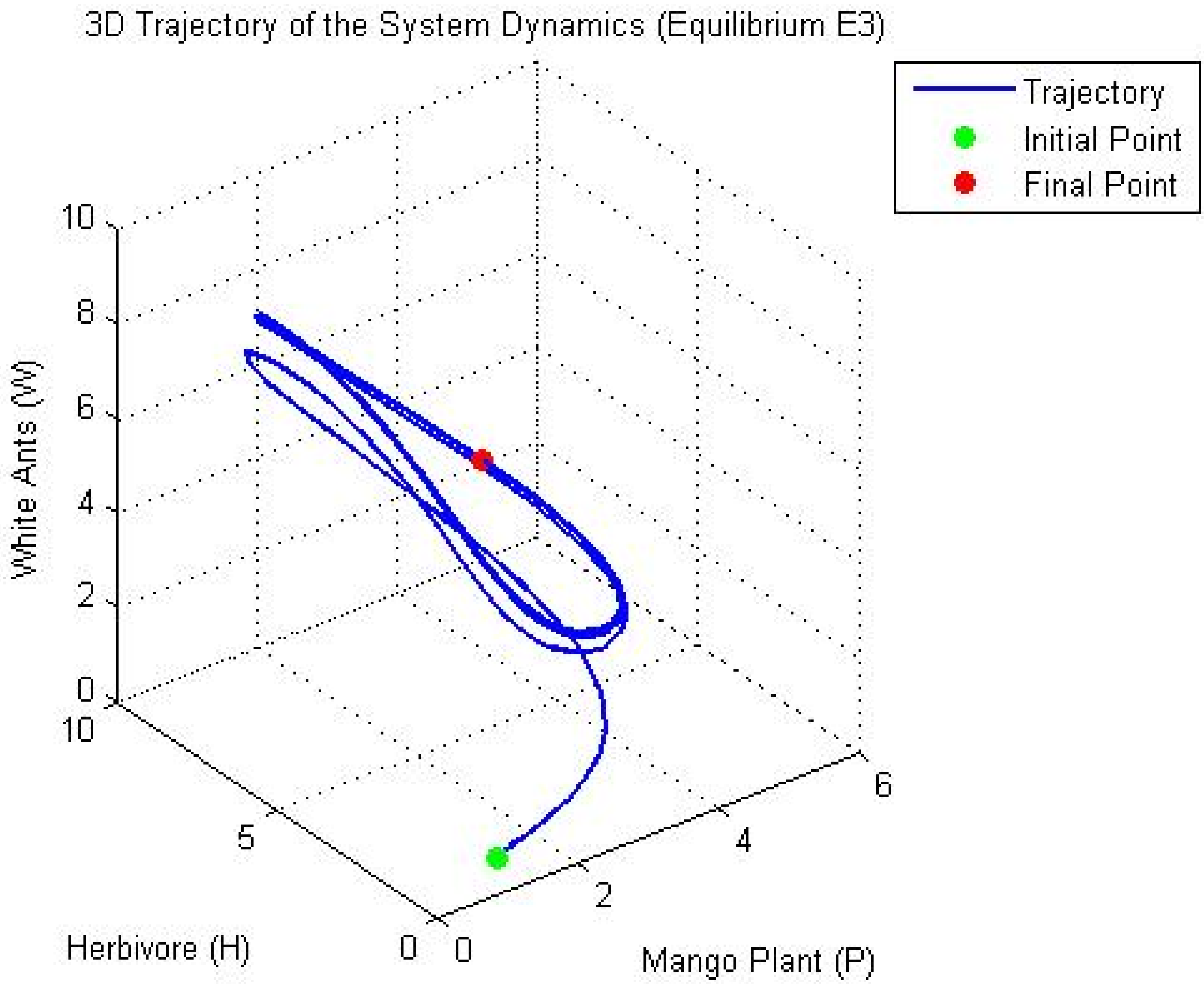
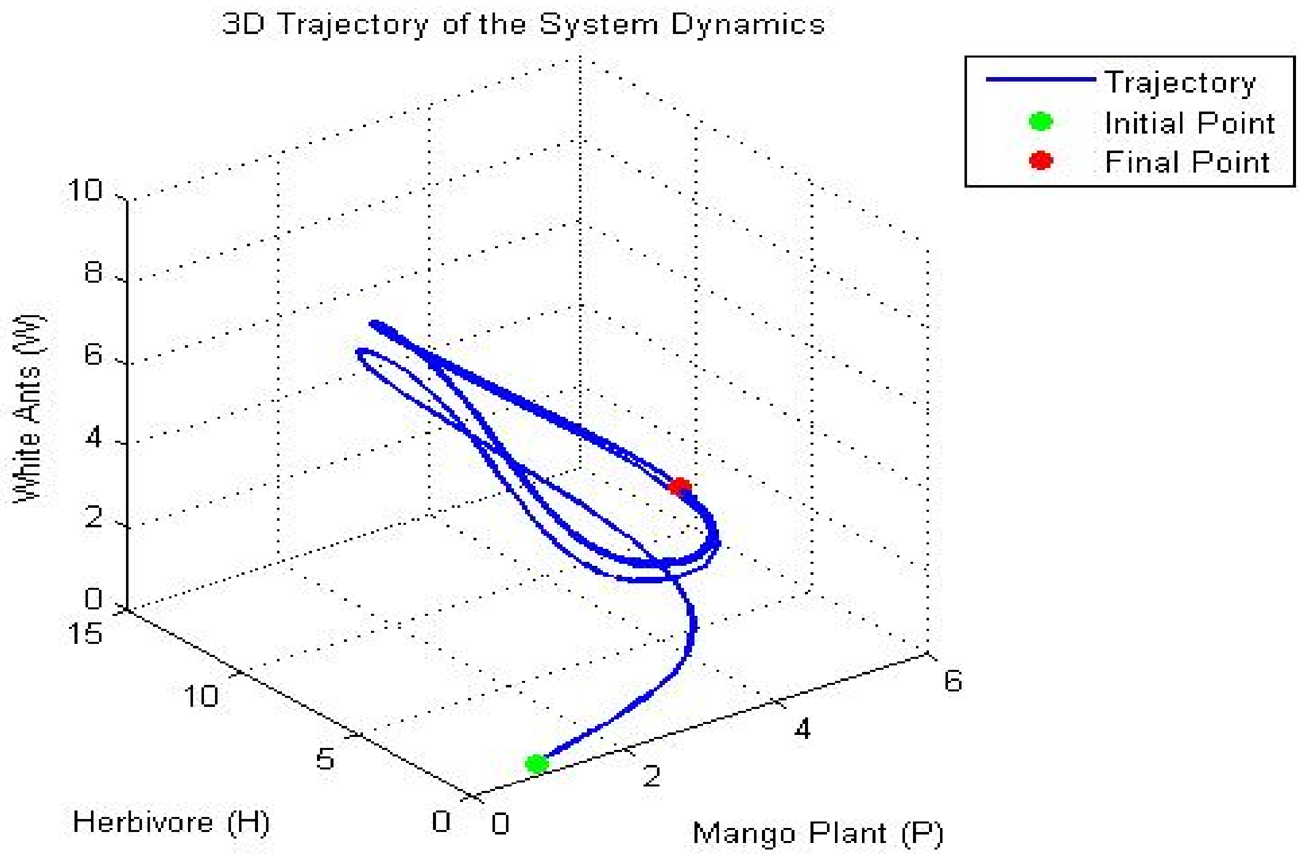


Figure 4: **Graph for** E_2


3D Trajectory of the System Dynamics (Equilibrium E3)
Trajectory
Initial Point
Final Point
White Ants (W)
Herbivore (H)
Mango Plant (P)
0
2
4
6
8
10
0
5
10
0
2
4
6


3D Trajectory of the System Dynamics
Trajectory
Initial Point
Final Point
White Ants (W)
Herbivore (H)
Mango Plant (P)
0
2
4
6
8
10
0
5
10
15
0
2
4
6

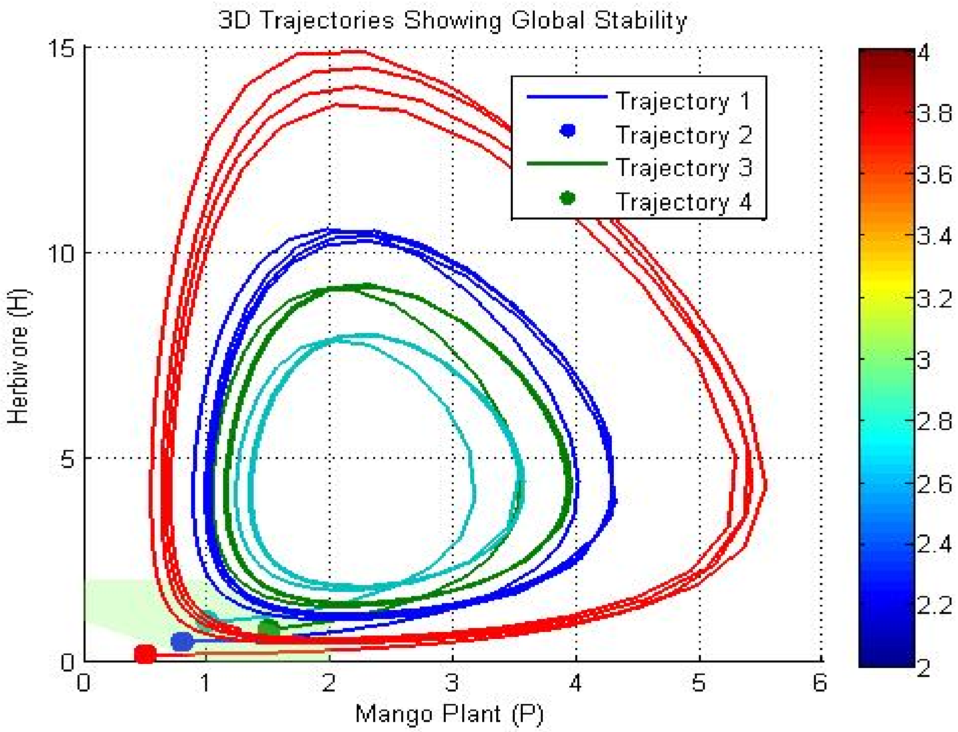


Figure 6: **The function V(P,H,W)=P+H+W is visualized as a translucent surface to show its role as a decreasing function guiding the trajectories toward equilibrium.**

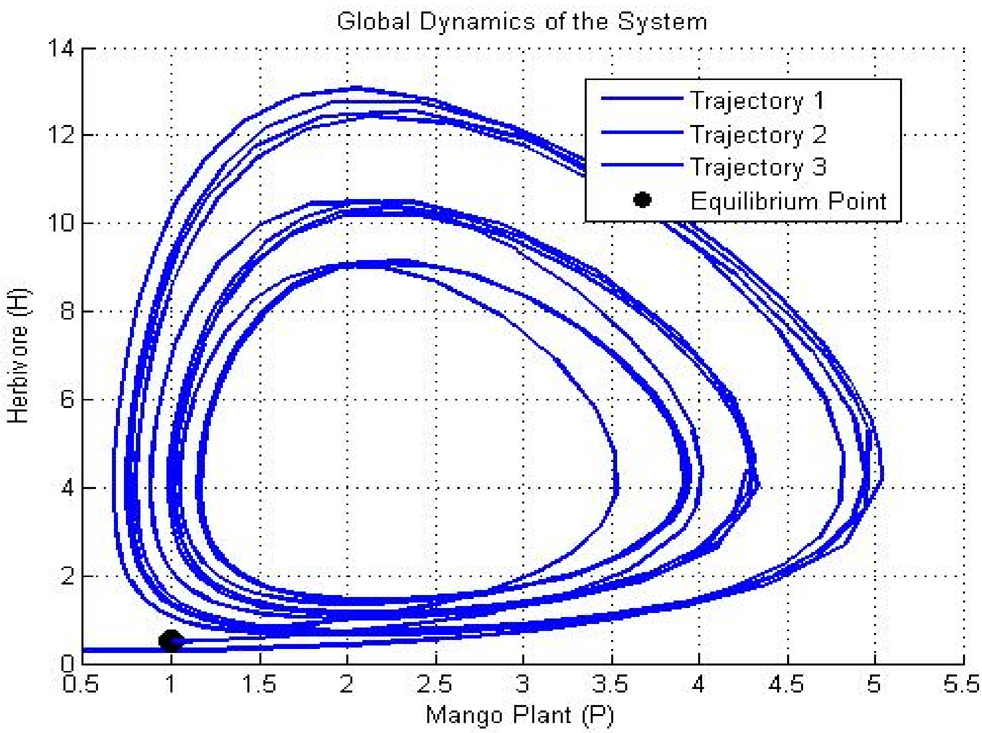


Figure 7: **The convergence of all trajectories toward the equilibrium point supports the global stability derived from the Lyapunov function.**