SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY BOREL DISTRIBUTION

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Abstract: The target of this paper is to define the operator of q- derivative based upon the Borel distribution and by using this operator, we obtain the coefficient bounds, inclusion relations, extreme points and some more properties of defined class.

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1. 1 Inroduction.

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk U = {z : |z| < 1} and normalized by $f(0) = 0, f'(0) = 1$. Let S be the subclass of A consisting of univalent functions $f(z)$ of the form (1.1). Further denote by T the subclass of A consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, (a_n \geq 0) \tag{1.2}$$

introduced and studied by Silverman [7].

For $g(z)=z+\sum_{n=2}^{\infty}b_n z^n$, the Hadamard product (or convolutions) of f and g is defined by

$$(f*g)\,z=z+a_n b_n z^n,\ z\in U \tag{1.3}$$

The elementary distribution such as the Poisson, the Pascal, the Logarithmic, the Binomial have been partially studied in the Geometric Function Theory from a theoretical point of view (see [1,2,5,6].

A discrete random variable x is said to have a Borel distribution if it takes the values $1,2,3,\cdots$ with the probabilities $\frac{e^{-\lambda}}{1!},\frac{2\lambda e^{-2\lambda}}{2!},\frac{9\lambda^2 e^{-3\lambda}}{3!},\cdots$, respectively, where λ is called the parameter. Very recently, Wanas and Khuttar [9] introduced the Borel distribution (BD) whose probability mass function is

$$P(x=\varrho)=\frac{(\varrho\lambda)^{\varrho-1}e^{-\lambda\varrho}}{\varrho!},\varrho=1,2,3,\cdots$$

Wanas and Khuttar [9] introduced a series $M(\lambda;z)$ whose coefficients are probabilities of the Borel Distribution (BD)

$$\begin{aligned} M(\lambda;z) &= z+\sum_{k=2}^{\infty}\frac{[\lambda(k-1)]^{k-2}e^{-\lambda(k-1)}}{(k-1)!}z^k, \quad (0<\lambda\le 1) \\ &= z+\sum_{k=2}^{\infty}\sigma_k(\lambda)z^k, (0<\lambda\le 1), \end{aligned} \tag{1.4}$$

where

$$\sigma_k(\lambda)=\frac{[\lambda(k-1)]^{k-2}e^{-\lambda(k-1)}}{(k-1)!}.$$

We define a linear operator $\mathfrak{B}(\lambda;z)f\colon A\to A$ as follows:

$$\begin{aligned} \mathfrak{B}(\lambda;z)f(z) &= M(\lambda;z)*f(z) \\ &= z+\sum_{k=2}^{\infty}\frac{[\lambda(k-1)]^{k-2}e^{-\lambda(k-1)}}{(k-1)!}a_k z^k,\ (0<\lambda\le 1). \end{aligned}$$

Srivastava [8] made use of various operators of q- calculus and fractional q- calculus and recalling the definition and notations. The q- shifted factorial is defined for $\lambda, q \in \mathbb{C}$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup 0$ as follows:

$$(\lambda; q)_k = \begin{cases} 1, & \text{for } k = 0, \\ (1-\lambda)(1-\lambda q)\cdot(1-\lambda q^{k-1}), & \text{for } k \in \mathbb{N}. \end{cases}$$

By using the q-gamma function $\Gamma_q(z)$, we get

$$\left(q^\lambda; q\right)_k = \frac{(1-q)^k \Gamma_q(\lambda+k)}{\Gamma_q(\lambda)} \quad (k \in \mathbb{N}_0),$$

where (see [8])

$$\Gamma_q(z) = (1-q)^{1-z} \frac{(q;q)_\infty}{(q^z;q)_\infty} \quad (|q| < 1).$$

Also, we note that

$$(\lambda; q)_\infty = \prod_{k=0}^{\infty} (1-\lambda q^k) \quad (|q| < 1)$$

and, the q- gamma function $\Gamma_q(z)$ is known

$$\Gamma_q(z+1) = [z]_q \Gamma_q(z),$$

where$[k]_q$ denotes the basic q- number defined as follows:

$$[k]_q = \begin{cases} \frac{1-q^k}{1-q}, & \text{for } k \in \mathbb{C}, \\ 1 + \sum_{j=1}^{k-1} q^j, & \text{for } k \in \mathbb{N}. \end{cases} \tag{1.5}$$

Using the definition formula (1.5) we have the next two products:

(i) For any non-negative integer k, the q-shifted factorial is given by

$$[k]_q! := \begin{cases} 1, & \text{for } k = 0, \\ \prod_{n=1}^{k} [n]_q, & \text{for } k \in \mathbb{N}. \end{cases}$$

(ii) For any positive number r, the q- generalized Poccammer symbol is defined by

$$[r]_{q,k} := \begin{cases} 1, & \text{for } k = 0, \\ \prod_{n=r}^{r+k-1} [n]_q, & \text{for } k \in \mathbb{N}. \end{cases}$$

It is known in terms of the classical (Euler's) gamma function $\Gamma(z)$, that

$$\Gamma_q(z) \to \Gamma(z) \quad \text{as} q \to 1^-$$

Also, we observe that

$$\lim_{q\to 1^-} \left\{ \frac{(q^\lambda; q)_k}{(1-q)^k} \right\} = (\lambda)_k.$$

For $0 < q < 1$, the q- derivative operator [8] (see also [9]) for $\mathfrak{B}(\lambda; z)f$ is defined by

$$\begin{aligned} D_q\big(\mathfrak{B}(\lambda; z)f(z)\big) \quad &:= \frac{\mathfrak{B}(\lambda; z)f(z) - \mathfrak{B}(\lambda; z)f(qz)}{z(1-q)} \\ &= 1 + \sum_{k=2}^{\infty} [k]_q \frac{[\lambda(k-1)]^{k-2} e^{-\lambda(k-1)}}{(k-1)!} a_k z^{k-1}, \quad (0 < \lambda \leq 1,\ z \in E), \end{aligned}$$

where

$$[k]_q := \frac{1-q^k}{1-q} = 1 + \sum_{j=1}^{k-1} q^j, \quad [0, q] := 0.$$

For $\vartheta > -1$ and $0 < q < 1$, we defined the linear operator $\mathfrak{B}_\lambda^{\vartheta,q} f\colon A \to A$ by

$$\mathfrak{B}_\lambda^{\vartheta,q} f(z) * N_{q,\vartheta+1}(z) = zD_q\big(\mathfrak{B}(\lambda; z)f(z)\big), \quad z \in E,$$

where the function $N_{q,\vartheta+1}$is given by

$$N_{q,\vartheta+1(z)} := z + \sum_{k=2}^{\infty} \frac{[\vartheta+1]_{q,k-1}}{[k-1]_q!} z^k, \quad z \in E.$$

A simple computation shows that

$$\begin{aligned}\mathfrak{B}_\lambda^{\vartheta,q} f(z): \quad &= z + \sum_{k=2}^{\infty} \frac{[k]_q!\,[\lambda(k-1)]^{k-2} e^{-\lambda(k-1)}}{[\vartheta+1]_{q,k-1}(k-1)!} a_k z^k \\ &= z + \sum_{k=2}^{\infty} B(\mathrm{k})\ a_k z^k, \qquad (1.6)\end{aligned}$$

where

$$B(\mathrm{k}) = \frac{[k]_q!\,[\lambda(k-1)]^{k-2} e^{-\lambda(k-1)}}{[\vartheta+1]_{q,k-1}(k-1)!} \qquad (1.7)$$

and$0 < \lambda \leq 1, \vartheta > -1, 0 < q < 1, z \in E$.

From the definition relation (1.6), we can easily verify that the next relations hold for all $f \in A$:

$$(i)\ \ [\vartheta+1]_q \mathfrak{B}_\lambda^{\vartheta,q} f(z) = [\vartheta]_q \mathfrak{B}_\lambda^{\vartheta+1,q} f(z) + q^{\vartheta} z D_q \left(\mathfrak{B}_\lambda^{\vartheta+1,q} f(z)\right), \quad z \in E$$

$$(ii)\ \ R_\lambda^{\vartheta} f(z) := \lim_{q \to 1^-} \mathfrak{B}_\lambda^{\vartheta,q} f(z) = z + \sum_{k=2}^{\infty} \frac{k[\lambda(k-1)]^{k-2} e^{-\lambda(k-1)}}{(\vartheta+1)_{k-1}} a_k z^k, \ \ z \in E.$$

Now using above differential operator, we define the following subclass of *T*.

Definition 1.1. Let $T_q(\alpha, \beta, \vartheta, \lambda)$ be the subclass of *T* consisting of functions which satisfy the conditions

$$\left\{\frac{z D_q\left(\mathfrak{B}_\lambda^{\vartheta,q} f(z)\right)}{\beta z D_q\left(\mathfrak{B}_\lambda^{\vartheta,q} f(z)\right) + (1-\beta)\mathfrak{B}_\lambda^{\vartheta,q} f(z)}\right\} > \alpha, \qquad (1.8)$$

for some α, β $(0 \leq \alpha, \beta < 1), 0 < \lambda \leq 1, \vartheta > -1$ and 0 < q <1.

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2 Coefficient bounds

Theorem 2.1. A function f(z) defined by (1.8) is in the class $T_q(\alpha, \beta, \vartheta, \lambda)$. if and only if

$$\sum_{k=2}^{\infty} B(k)a_k\big[(1-\alpha\beta)[k]_q + \alpha\beta - \alpha\big] < 1-\alpha, \qquad (2.1)$$

where, $B(\mathrm{k})$ is defined in (1.7).

Proof. Suppose $f \in T_q(\alpha, \beta, \vartheta, \lambda)$. Then

$$\Re\left\{\frac{zD_q\left(\mathfrak{B}_\lambda^{\vartheta,q}f(z)\right)}{\beta zD_q\left(\mathfrak{B}_\lambda^{\vartheta,q}f(z)\right) + (1-\beta)\mathfrak{B}_\lambda^{\vartheta,q}f(z)}\right\} > \alpha,$$

$$\Re\left\{\frac{z - \sum_{k=2}^{\infty} B(k)[k]_q a_k z^k}{\beta\left[z - \sum_{k=2}^{\infty} B(k)[k]_q a_k z^k\right] + (1-\beta)\left[z - \sum_{k=2}^{\infty} B(k)a_k z^k\right]}\right\} > \alpha,$$

$$\Re\left\{\frac{z - \sum_{k=2}^{\infty} B(k)[k]_q a_k z^k}{z - \sum_{k=2}^{\infty} B(k)a_k z^k\left[\beta\left([k]_q - 1\right) + 1\right]}\right\} > \alpha.$$

Letting $z \to 1$, we get,

$$1 - \sum_{k=2}^{\infty} B(k)[k]_q a_k > \alpha\left\{1 - \sum_{k=2}^{\infty} B(k)a_k\left[\beta\left([k]_q - 1\right) + 1\right]\right\}.$$

Equivalently we have,

$$\sum_{k=2}^{\infty} B(k)[k]_q a_k - \alpha\left\{\sum_{k=2}^{\infty} B(k)a_k\left[\beta\left([k]_q - 1\right) + 1\right]\right\} < (1-\alpha)$$

which implies

$$\sum_{k=2}^{\infty} B(k)a_k\left[(1-\alpha\beta)[k]_q + \alpha\beta - \alpha\right] < (1-\alpha).$$

Conversely, assume that (2.1) is be true. To show that $f \in T_q(\alpha,\beta,\vartheta,\lambda)$, we need to prove the inequality (1.8). For this consider

$$\left|\left\{\frac{zD_q\left(\mathfrak{B}_\lambda^{\vartheta,q} f(z)\right)}{\beta zD_q\left(\mathfrak{B}_\lambda^{\vartheta,q} f(z)\right)+(1-\beta)\mathfrak{B}_\lambda^{\vartheta,q} f(z)}\right\}-1\right|<1-\alpha.$$

But

$$\left|\left\{\frac{z-\sum_{k=2}^{\infty} B(k)[k]_q a_k z^k}{z-\sum_{k=2}^{\infty} B(k)a_k z^k\left[\beta\left([k]_q-1\right)+1\right]}\right\}-1\right|$$

$$=\left|\frac{\sum_{k=2}^{\infty} B(k)a_k\left([k]_q-1\right)(\beta-1)z^k}{z-\sum_{k=2}^{\infty} B(k)a_k\left[\beta\left([k]_q-1\right)+1\right]z^k}\right|$$

$$\leq\frac{\sum_{k=2}^{\infty} B(k)a_k\left([k]_q-1\right)(\beta-1)|z^k|}{|z|-\sum_{k=2}^{\infty} B(k)a_k\left[\beta\left([k]_q-1\right)+1\right]|z^k|}$$

$$\leq\frac{\sum_{k=2}^{\infty} B(k)a_k\left([k]_q-1\right)(\beta-1)}{1-\sum_{k=2}^{\infty} B(k)a_k\left[\beta\left([k]_q-1\right)+1\right]}$$

The last expression is bounded above by $1-\alpha$ if

$$\sum_{k=2}^{\infty} B(k)a_k\left([k]_q-1\right)(\beta-1)$$
$$\leq(1-\alpha)\left(1-\sum_{k=2}^{\infty} B(k)a_k\left[\beta\left([k]_q-1\right)+1\right]\right)$$

or

$$\sum_{k=2}^{\infty} B(k)a_k\left[(1-\alpha\beta)[k]_q+\alpha\beta-\alpha\right]<1-\alpha,$$

which is true by hypothesis. This completes the assertion of Theorem 2.1.

Corollary 2.1. If $f \in T_q(\alpha,\beta,\vartheta,\lambda)$. then

$$|a_k| \leq \frac{1-\alpha}{B(k)\left[(1-\alpha\beta)[k]_q + \alpha\beta - \alpha\right]}.$$

Theorem 2.2. Let $0 \leq \alpha < 1, 0 \leq \beta_1 \leq \beta_2 < 1$,, then $T_q(\alpha, \beta_1, \vartheta, \lambda) \subset T_q(\alpha, \beta_2, \vartheta, \lambda)$.

Proof. For $f(z) \in T_q(\alpha, \beta_2, \vartheta, \lambda)$. We have,

$$\sum_{k=2}^{\infty} B(k) a_k \left[(1-\alpha\beta_2)[k]_q + \alpha\beta_2 - \alpha\right]$$
$$\leq \sum_{k=2}^{\infty} B(k) a_k \left[(1-\alpha\beta_1)[k]_q + \alpha\beta_1 - \alpha\right] < 1-\alpha.$$

Hence $f(z) \in T_q(\alpha, \beta_1, \vartheta, \lambda)$.

3 Extreme points and Closure property

Theorem 3.1. Let $f \in T_q(\alpha, \beta, \vartheta, \lambda)$. Define $f_1(z) = z$ and

$$f_k(z) = z + \frac{1-\alpha}{B(k)\left[(1-\alpha\beta)[k]_q + \alpha\beta - \alpha\right]} z^k, k = 2,3,\cdots,$$

for some $\alpha, \beta (0 \leq \beta < 1)$ and $z \in E$. Then $f \in T_q(\alpha, \beta, \vartheta, \lambda)$ if and only if $f(\mathrm{z})$ can be expressed as $f(\mathrm{z}) = \sum_{\mathrm{k}=1}^{\infty} \mu_{\mathrm{k}} f_k(\mathrm{z})$ where $\mu_{\mathrm{k}} \geq 0$ and $\sum_{\mathrm{k}=1}^{\infty} \mu_{\mathrm{k}} = 1$.

Proof. If $f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z)$ with $\sum_{k=1}^{\infty} \mu_k = 1, \mu_k \geq 0$, then

$$\sum_{k=2}^{\infty} \frac{B(k)\left[(1-\alpha\beta)[k]_q + \alpha\beta - \alpha\right]\mu_k}{B(k)\left[(1-\alpha\beta)[k]_q + \alpha\beta - \alpha\right]}(1-\alpha) \sum_{k=2}^{\infty} \mu_k (1-\alpha)$$

$$= (1-\mu_1)(1-\alpha) \leq (1-\alpha).$$

Hence $f \in T_q(\alpha, \beta, \vartheta, \lambda)$.

Conversely, let $f(z) = z - \sum_{\mathrm{k}=2}^{\infty} \mathrm{a_k} z^{\mathrm{k}} \in T_q(\alpha, \beta, \vartheta, \lambda)$. Define

$$\mu_k = \frac{B(k)\left[(1-\alpha\beta)[k]_q + \alpha\beta - \alpha\right]|a_k|}{(1-\alpha)}, k = 2,3,\cdots,$$

and define $\mu_1 = 1 - \sum_{\mathrm{k}=2}^{\infty}\mu_{\mathrm{k}}$. From Theorem (2.2), $\sum_{\mathrm{k}=2}^{\infty}\mu_{\mathrm{k}} \leq 1$ and hence $\mu_1 \geq 0$. Since

$\mu_k f_k(z) = \mu_k f(z) + a_k z^k$, $\sum_{k=1}^{\infty}\mu_k f_k(z) = z - \sum_{k=2}^{\infty} a_k z^k = f(z)$.

Theorem 3.2. The class $T_q(\alpha,\beta,\vartheta,\lambda)$. is closed under convex linear combination.

Proof. Let $f(z), g(z) \in T_q(\alpha,\beta,\vartheta,\lambda)$ and let

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, g(z) = z - \sum_{k=2}^{\infty} b_k z^k.$$

For η such that $0 \leq \eta \leq 1$, it suffices to show that the function defined by

$h(z) = (1-\eta)f(z) + \eta g(z), z \in E$ belongs to $T_q(\alpha,\beta,\vartheta,\lambda)$. Now

$$h(z) = z - \sum_{k=2}^{\infty} [(1-\eta)a_k + \eta b_k]z^k.$$

Applying Theorem 2.2, to $f(z), g(z) \in T_q(\alpha,\beta,\vartheta,\lambda)$. We have

$$\sum_{k=2}^{\infty} \mathrm{B(k)}\big[(1-\alpha\beta)[\mathrm{k}]_{\mathrm{q}} + \alpha\beta - \alpha\big][(1-\eta)a_k + \eta b_k]$$
$$= (1-\eta)\sum_{\mathrm{k}=2}^{\infty} \mathrm{B(k)}\big[(1-\alpha\beta)[\mathrm{k}]_{\mathrm{q}} + \alpha\beta - \alpha\big]a_{\mathrm{k}}$$
$$+\eta\sum_{\mathrm{k}=2}^{\infty} \mathrm{B(k)}\big[(1-\alpha\beta)[\mathrm{k}]_{\mathrm{q}} + \alpha\beta - \alpha\big]b_k$$
$$\leq (1-\eta)(1-\alpha) + \eta(1-\alpha) = (1-\alpha).$$

This implies that $\mathrm{h}(z) \in T_q(\alpha,\beta,\vartheta,\lambda)$.

Corollary 3.1. If $f_1(z), f_2(z)$ are in $T_q(\alpha,\beta,\vartheta,\lambda)$ then the function defined by

$g(z) = \frac{1}{2}[f_1(z) + f_2(z)]$ is also in $T_q(\alpha,\beta,\vartheta,\lambda)$.

Theorem 3.3. Let for $j = 1,2,\cdots,k$, $f_j(\mathrm{z}) = z - \sum_{\mathrm{k}=2}^{\infty} a_{k,j} z^k \in T_q(\alpha,\beta,\vartheta,\lambda)$ and $0 < \beta_j < 1$

such that $\sum_{j=1}^{k}\beta_j = 1$, then the function $F(z)$ defined by $F(z) = \sum_{j=1}^{k}\beta_j f_j(z)$ is also in $T_q(\alpha,\beta,\vartheta,\lambda)$.

Proof. For each $j \in \{1,2,3,\cdots,k\}$ we obtain

$$\sum_{k=2}^{\infty} B(k)\big[(1-\alpha\beta)[k]_q + \alpha\beta - \alpha\big]|a_k| < (1-\alpha).$$

$$F(z) = \sum_{j=1}^{k} \beta_j \left(z - \sum_{k=2}^{\infty} a_{k,j} z^k\right) = z - \sum_{k=2}^{\infty} \left(\sum_{j=1}^{k} \beta_j a_{k,j}\right) z^k$$

$$\sum_{k=2}^{\infty} B(k)\big[(1-\alpha\beta)[k]_q + \alpha\beta - \alpha\big]\left[\sum_{j=1}^{k} \beta_j a_{k,j}\right]$$

$$= \sum_{j=1}^{k} \beta_j \left[\sum_{k=2}^{\infty} B(k)\big[(1-\alpha\beta)[k]_q + \alpha\beta - \alpha\big]\right]$$

$$< \sum_{j=1}^{k} \beta_j (1-\alpha) < (1-\alpha).$$

Therefore $\mathrm{F}(z) \in T_q(\alpha,\beta,\vartheta,\lambda)$.

Conclusion: The study has introduced and analyzed a specific subclass of analytic functions defined by the q-analogue differential operator. The main findings include new properties and characterizations that distinguish this subclass from those defined by classical differential operators.These results offer valuable insights into the structure and behavior of analytic functions within the framework of q-calculus, presenting potential applications in various fields of mathematical and physical sciences.

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