PBIB-Designs arising from subgraphs of Hypercube networks

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Abstract

Properties and behaviour of hypercube networks, due to their highly symmetric nature, robustness and reliability have always been an interesting concept for network theorists. In this paper, we have obtained new class of PBIB-designs arising from hypercube networks Q_n where blocks are subgraphs, which are also hypercube networks Q_m, of Q_n. These newly constructed PBIB-designs belong to the sparse class of designs having large number of associate classes. Also we have given a new construction of strongly regular graphs with $d = q$ from Q_n with vertices as hyperplanes Q_{n-1} of Q_n.

Keywords. *Interconnection network, Hypercube network, PBIB-design, Group-divisible design, Strongly regular graph.*

Mathematics Subject Classification [2010] Primary 05C12, 05E30, 05B05

1 Introduction

A system, according to Hayes [13], is a collection of objects called components connected to form a coherent entity that is entitled to perform a well-defined function or a purpose. Performances of components and the manner in which these components are interconnected determines the performance of a system. Some of the examples are computer systems, multiple processor systems, electronic circuits, computer networks, pipeline systems, transportation systems to name a few. The pattern in which components are connected is called an interconnection network of the system. Interconnection networks can be modeled by a simple graph where the components are denoted as vertices and edges are the physical communication links that connect components. Hypercube network is an important interconnection network [27].

Hypercube (or binary n-cube multiprocessor) structure Q_n represents a loosely coupled system made up of 2^n processors interconnected in an n-dimensional binary cube. Therefore, each vertex here is said to contain a processor and each processor can communicate to any adjacent processor through n direct communication links. There are two distinct n-bit binary addresses which can be assigned to processors and addresses of any two adjacent processors that differ in exactly one bit position from each other [28]. Routing messages through an n-cube structure may take one to n links depending on the addresses of source and destination processors. For example, in a 3-cube structure, processor with address 000 may communicate to another processor having address 111 through any six distinct paths. They are 000-100-110-111, 000-100-101-111, 000-010-110-111, 000-010-011-111, 000-001-101-111 or 000-001-011-111. A routing procedure is designed by determining the exclusive-OR of the source vertex address with the destination vertex address and depending on that, message is transmitted along any one of the paths. For more details about routing of messages under certain conditions in hypercube network, readers can refer to [11].

Interconnection network is the most crucial part in parallel processing system. Owing to its interesting features, hypercube topology is considered a suitable choice for parallel processing. Authors in [21] have presented two innovative configurations of interconnection networks based on fractal Sierpinski graphs and a hypercube. They had considered different topologies of Sierpinski graphs and compared it with properties of hypercube networks. Though Sierpinski graphs have higher degree, high bisection width, lesser number of vertices than a hypercube network of same dimension, a major drawback is that a Sierpinski graph has diameter and average distance more than the hypercube graph. Since distance and diameter of an interconnection network is considered as an important metric that reflects its performance in transmission of messages, a hypercube network becomes a promising choice for parallel processing over other interconnection networks. Other properties that aid in hypercube network being used as an interconnection network is due to its efficient self-routing for effective communication and for its reliability, robustness and fault tolerance that can be measured using many graph parameters [12].

A hypercube of dimension n has regular degree n which makes it complicated or practically impossible to construct hypercube machines of higher dimension due to its expensive cost. Another disadvantage is exponentially growing number of vertices 2^n with increase in n value. Therefore, while designing a hypercube network, it is customary to reduce the number of interconnections, without actually compromising on robustness of the network. Hence, it becomes imperative to consider substructures based on hypercube networks that carry many important properties of the original network hereditarily. One such method is reducing dimension of a hypercube to obtain hyperplanes. And the other method is to consider some particular substructures or subgraphs of a hypercube. These reasons led to introduction of special subgraphs of hypercubes such as crossed cube, twisted cube, Fibonacci cube, hierarchical hypercube etc. [18], [23], [12]. Hierarchical hypercube network, ($N = 2^n,\ n = 2^m + m, m \geq 2$) [23] was proposed to achieve advantages in efficient communication and symmetric topology like hypercube, while being able to connect many significant number of vertices with lower connection cost. Hierarchical hypercube network is more suitable for scalable multiprocessors than a hypercube network. In this paper, we have considered lower dimensional hypercubes which are substructures of a hypercube network and obtained combinatorially consistent properties studied through PBIB-designs arising from them. Hence they ensure the hereditary propagation of properties of a hypercube into substructures.

Hypercubes and their subgraphs have been of great importance with interest derived from both their pure mathematical structures and numerous applications in coding theory, parallel computing, computer architecture, biology, chemistry, finite automata, electrical circuits, genetics, to name a few. In chemistry it finds wide range of applications in enumeration of isomers, isomerization reactions, visualization and computer graphics, chirality, protein-protein interactions, intrinsically disordered proteins etc. [5], [6]. The automorphism groups of hypercubes find applications in enumerative combinatorics, nuclear spin statistics, weakly-bound non-rigid water clusters, non-rigid molecules and proteomics. They have also been connected to Goldbach conjecture, Fermat's last theorem, Erdös discrepancy conjecture, modern multi-dimensional representation of time measures, quantum similarity measures, biochemical imaging, multi-dimensional imaging, classification of large data, Quantitative Shape-Activity Relations (QShAR) etc. [17], [19], [20]. In [1], authors have found generating functions for combinatorial enumeration of colorings of different hyperplanes, especially vertices of hypercubes which have been the topic of several studies for the past two centuries.

Combinatorial design theory is a part of combinatorics that deals with existence, construction and properties of systems of finite sets whose arrangements satisfy certain conditions. Balanced incomplete block (BIB)-designs and partially balanced incomplete block (PBIB)-designs are two major subfields finding a wide range of applications. PBIB-designs have a long history and have been extensively used in agriculture and industrial experiments. Over the years, different block designs have been constructed from graphs by taking blocks as certain subsets of the vertex set of a graph. Some remarkable PBIB-designs obtained from a graph are due to Ionin and Shrikhande [16], Walikar et al. [26] and Huilgol et al. [14], [15], where blocks are certain vertex subsets of a graph. Group divisible design (GDD), is an important concept in statistical design of experiments. For the existence of such a design, there are some restrictions on its parameters. So, given a set of parameters satisfying all the necessary constructions, it is yet a problem to say whether such a design exists or not. In [8], authors have developed an algorithm which looks for the existence of a group divisible design with certain restrictions. Many researchers have checked construction and existence of group divisible designs satisfying certain properties [10], [25].

The use of 2-associate class PBIB-designs is common in experimental work. However, PBIB-designs with more than three associate classes are not widely used because of the complicated nature of analysis and construction involved. In literature, only few results can be found on construction of designs with more than four-associate classes. Therefore, in this paper, we have constructed PBIB-designs arising from n-dimensional hypercube Q_n where blocks are vertices of hypercubes of lower dimensions with n-associate classes. We have obtained a new construction of strongly regular graph G from hypercube Q_n where vertices are hyperplanes of Q_n and any two vertices are adjacent if the corresponding hyperplanes share an edge and these strongly regular graphs with parameters (v', d, p, q) exhibit the property $d = q$.

2 Preliminaries

First we list out some basic definitions pertaining to graph theory and design theory. Undefined graph theoretical terms are used in the sense of Buckley and Harary [4] and undefined design theoretical terms are in the sense of Colbourn et al. [7].

Definition 2.1. *[3] A regular graph on v' vertices and degree d is called a strongly regular graph with parameters (v', d, p, q) if any two adjacent vertices have p common neighbours and any two non-adjacent vertices have q common neighbours and these numbers are independent of the pair of vertices chosen.*

Remark 1. *[3] All connected strongly regular graphs have diameter 2.*

Remark 2. *[4] If G is a strongly regular graph with parameters (v', d, p, q), then $(v' - d - 1)q = d(d - 1 - p)$.*

Definition 2.2. *[22] A balanced incomplete block (BIB)-design is a set of v elements arranged in b blocks of k elements each in such a way that each element occurs in exactly r blocks and every pair of unordered elements in λ blocks. The combinatorial configuration so obtained is called a (v, b, r, k, λ)-design. A BIB-design satisfies the following conditions.*
(i) $vr = bk$ (ii) $\lambda(v - 1) = r(k - 1)$ (iii) $b \geq v$.

Definition 2.3. *[24] Given a set $\{1, 2, 3, \ldots, v\}$ of v elements, a relation satisfying the following conditions is said to be an association scheme with m classes.*

- *Any two elements α and β are i^{th} associates for some i with $1 \leq i \leq m$ and this relation of being i^{th} associates is symmetric.*
- *The number of i^{th} associates of each element is n_i.*
- *If α and β are two elements which are i^{th} associates, then the number of elements which are j^{th} associates of α and k^{th} associates of β is p^i_{jk} and is independent of the pair of i^{th} associates α and β.*

Definition 2.4. *[2] Consider a set $V = \{1, 2, \ldots, v\}$ and an association scheme with m classes on V. A partially balanced incomplete block (PBIB)-design represented as $(v, b, r, k, \lambda_1, \ldots, \lambda_m)$ is a collection of b subsets of V called blocks, each of them containing k elements $(k < v)$ such that every element occurs in r blocks and any two elements α and β which are i^{th} associates occur together in λ_i blocks, numbers λ_i being independent of the choice of pairs α and β.*

The numbers $v,\ b,\ r,\ k,\ \lambda_i\ (i = 1, 2, \ldots, m)$ *are called parameters of first kind and* $n_i' s$ *and* p^i_{jk} *are called parameters of second kind.*

Definition 2.5. *[9] A group divisible design* (GDD) *with parameters* $(v = v_1 + v_2 + \cdots + v_m, b, \lambda_1^*, \lambda_2^*; m, n)$ *is an ordered pair* $(\mathcal{V}, \mathcal{B})$ *where* $\mathcal{V}$ *is a* v*-set of elements partitioned into* m *classes (containing* n *elements each) called groups, with sizes* $v_1, v_2, \ldots, v_m$ *and* $\mathcal{B}$ *is a collection of* k*-subsets of* $\mathcal{V}$*, called blocks, such that each pair of elements from the same group appear together in* λ_1^* *blocks and each pair of elements from distinct groups appear together in* λ_2^* *blocks.*

λ_1^* *and* λ_2^* *are called indices of the design.*

A group divisible design is a PBIB-design where the set of elements are partitioned into groups with two different associates. Elements occurring together in the same group are called first associates, and elements occurring in different groups are called second associates.

Note: In geometry, a hyperplane is a subspace whose dimension is one less than the ambient space. For example, if a space is 3-dimensional, then its hyperplanes are 2-dimensional planes. Similarly if a space is 2-dimensional, then its hyperplanes are 1-dimensional lines.

3 Results

In this section we give few important results that have been obtained by considering subgraphs of an n-dimensional hypercube Q_n. Before diving into results, let us view the structure of hypercubes of different dimensions and some well-known properties of hypercubes Q_n.

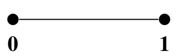

Figure 1: Q_1

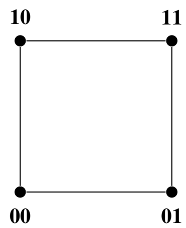

Figure 2: Q_2

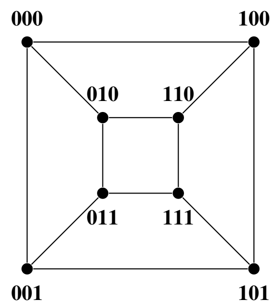

Figure 3: Q_3

Hypercube Q_n of dimension n is an n-regular graph on 2^n vertices and $2^{n-1}n$ edges, whose vertex set is the set of all n-dimensional boolean vectors in which two vectors are joined if and only if they differ in exactly one coordinate. Distance between two vertices is the number of coordinates that differ in their labels. We observe that $Q_1 = K_2$, a complete graph on two vertices, and $Q_n = Q_{n-1} \times K_2$, if $n \geq 2$, that is, cartesian product of Q_{n-1} and K_2. Note that hypercube of dimension n is a self-centered, distance degree regular and unique eccentric vertex graph of diameter n. Q_n decomposes into Hamiltonian cycles if n is even and a perfect matching and Hamiltonian cycles if n is odd.

Next we give a generalized result for parameters of PBIB-design obtained from hypercube Q_n, where we consider blocks to be m-dimensional hypercubes Q_m, which are subgraphs of Q_n for $2 \leq m < n$.

Theorem 3.1. *A partially balanced incomplete block (PBIB)-design exists with parameters* $(v, b, r, k, \lambda_1, \lambda_2, \ldots, \lambda_m, \lambda_{m+1}, \ldots, \lambda_n) = \left(2^n, 2^{n-m}\binom{n}{m}, \binom{n}{m}, 2^m, \frac{\binom{m}{1}\binom{n}{m}}{\binom{n}{1}}, \frac{\binom{m}{2}\binom{n}{m}}{\binom{n}{2}}, \ldots, 1, 0, \ldots, 0\right)$ *where blocks are subgraphs of* Q_n*, which are* m*-dimensional hypercubes* Q_m*, for* $2 \leq m < n$*.*

Proof. Let $Q(m, n)$, where $m < n$, denote the number of distinct hypercubes Q_m in Q_n. First let us count the number of Q_m at each vertex of Q_n. Since Q_n has regularity n, each set of m edges emanating from a vertex forms part of a Q_m. Thus, we get $\binom{n}{m}$ number of Q_m arising from each vertex of Q_n. Since there are 2^n vertices in Q_n, we get $2^n\binom{n}{m}$ number of Q_m in Q_n, but not all distinct as each Q_m is counted 2^m times corresponding to each vertex of Q_m. Hence, there are $2^{n-m}\binom{n}{m}$ number of distinct subgraphs which are m-dimensional hypercubes Q_m, in Q_n, giving the value of $Q(m, n)$ as $2^{n-m}\binom{n}{m}$.

Clearly, $\binom{n}{m}$ number of Q_m's arise from each vertex implying that each vertex appears in $\binom{n}{m}$ number of Q_m's which is the repetition number of design. As we consider blocks to be Q_m, vertices of each Q_m forms a block, hence, block size is 2^m.

To obtain the value of λ_1 we count number of Q_m's containing a pair of vertices which are at Hamming distance 1 from each other. Consider any arbitrary vertex, say x, in Q_n. Clearly x has n-distinct neighbours in Q_n. Let vertex, say y, be one of the neighbours of x. It is obvious that all these n neighbours of x do not appear together with x in any Q_m as regularity of Q_m is m and $m < n$. Hence, these n neighbours of x are distributed uniformly in r number of Q_m's containing vertex x symmetrically. Thus, there are $\frac{k \times r}{n}$ number of Q_m's containing pair of adjacent vertices, x and y.

In general, vertex x has $\binom{n}{m}$ number of vertices at distance m from x distributed uniformly in r Q_m's containing vertex x. Thus, number of Q_m's containing a pair of vertices which are at Hamming distance i from each other is $\lambda_i = \frac{\binom{m}{i}\binom{n}{m}}{\binom{n}{i}}$ for i, $1 \leq i \leq m < n$. When $i = m$, we get $\lambda_m = 1$. No pair of vertices at Hamming distance greater than m from each other appear together in Q_m as Q_m is a regular graph of diameter m. So $\lambda_i = 0$ for i, $m + 1 \leq i \leq n$.

Thus, considering blocks as hypercubes Q_m which are subgraphs of Q_n yields a partially balanced incomplete block (PBIB)-design with parameters $(v, b, r, k, \lambda_1, \lambda_2, \ldots, \lambda_m, \lambda_{m+1}, \ldots, \lambda_n) = \Big(2^n, 2^{n-m}\binom{n}{m}, \binom{n}{m}, 2^m, \frac{\binom{m}{1}\binom{n}{m}}{\binom{n}{1}},$

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$\frac{\binom{m}{2}\binom{n}{m}}{\binom{n}{2}}, \ldots, 1, 0, \ldots, 0\Big).$ □

Corollary 3.1. *There does not exist a group divisible design where blocks are subgraphs Q_m of hypercube Q_n, for $n \geq 3$ and $2 \leq m < n$.*

Proof. Grouping the vertices of Q_n into two groups, one containing vertices of even parity and the other containing vertices of odd parity, we see that for the design to be group divisible design, all λ_i should be equal when i is even (odd). This implies that any pair of vertices from the same group should occur in equal number of blocks which is a necessary requirement for a group divisible design to exist. But λ_i, $1 \leq i \leq n$, in Theorem 3.1 do not satisfy this condition. Hence, design obtained in Theorem 3.1 cannot form a group divisible design when $n \geq 3$. □

Illustration: Consider a 3-dimensional hypercube Q_3 as shown in Figure 3. Note that the vertices of Q_3 are labeled as follows: $\{000, 110, 101, 011, 100, 010, 001, 111\}$. The hyperplanes Q_2 that are subgraphs of Q_3 are $\{000, 100, 110, 010\}$, $\{000, 010, 011, 001\}$, $\{100, 110, 111, 101\}$, $\{101, 111, 011, 001\}$, $\{010, 110, 111, 011\}$ and $\{000, 100, 101, 001\}$. We see that each vertex appears in three Q_2's. Any pair of adjacent vertices appear together in 2 Q_2's and any pair of vertices which are at Hamming distance 2 from each other appear together in exactly one Q_2. Since the blocks are of dimension 2, no two vertices which are at Hamming distance 3 from each other appear together in any block. Considering the above six 2-dimensional hypercubes Q_2 as blocks, we get parameters of PBIB-design as $(v, b, r, k, \lambda_1, \lambda_2, \lambda_3)$ = $(8, 6, 3, 4, 2, 1, 0)$. Clearly, we cannot obtain a group divisible design from these blocks. The vertex set of Q_3 can be partitioned into two groups $\{000, 110, 101, 011\}$ and $\{100, 010, 001, 111\}$. The first group has even parity vertices including the vertex containing 0 in all coordinates and second group has odd parity vertices. Any two vertices from the same group appear together in one Q_2 but any two vertices from different groups, especially which are at Hamming distance 1 appear together in two blocks and a pair of vertices which are at Hamming distance 3 from each other do not appear together in any block. Hence, λ_2^* is not unique implying that group divisible design cannot be obtained by considering blocks as hyperplanes of Q_3.

Now, we give a construction for a graph G where vertices of G are hyperplanes of Q_n and any two vertices are adjacent if the corresponding hyperplanes share an edge. Theorem 3.2 proves that such graphs G obtained are strongly regular with parameters $(2n, 2n-2, 2n-4, 2n-2)$.

Theorem 3.2. *A strongly regular graph G having parameters (v', d, p, q) = $(2n, 2n-2, 2n-4, 2n-2)$ is obtained from n-dimensional hypercube Q_n where vertices of graph G are hyperplanes Q_{n-1}, and any two vertices are adjacent if the corresponding hyperplanes share an edge.*

Proof. We know that there are $2^{n-m}\binom{n}{m}$ number of hypercubes Q_m in Q_n. Taking m to be $n-1$, we count the number of hyperplanes in a hypercube Q_n. Thus, there are $2n$ number of hyperplanes Q_{n-1} in Q_n. Taking each

Q_{n-1} as a vertex, we form a graph G where any two vertices in G are adjacent if the corresponding hyperplanes share an edge. Consider a vertex, say, x in G. Clearly there is exactly one vertex, say, z in G whose corresponding hyperplane has all 2^{n-1} labeled vertices distinct from that in x, thus forming an eccentric vertex of x. This is true for all $2n$ vertices. Hence, G is unique eccentric vertex graph with regularity $2n-2$. Clearly G has diameter 2.

Consider a pair of adjacent vertices, say, x and y in G. Since G is a unique eccentric vertex graph, remaining $2n-4$ vertices are adjacent to both x and y. Let x and z be a pair of non-adjacent vertices. Clearly, they are eccentric vertices of each other and remaining $2n-2$ vertices are adjacent to both x and z. Since vertex x is arbitrary, the same argument holds for all other vertices of G. Hence, graph G is strongly regular with srg parameters (v', d, p, q) = $(2n, 2n-2, 2n-4, 2n-2)$. □

Illustration: Consider a 4-dimensional hypercube Q_4 shown in Figure 4. Vertices of Q_4 are 4-tuples with any two vertices adjacent if they are at Hamming distance 1 as shown. There are 8 distinct hyperplanes which can be obtained from Q_4 as subgraphs. Constructing the graph as mentioned in Theorem 3.2 yields a strongly regular graph G.

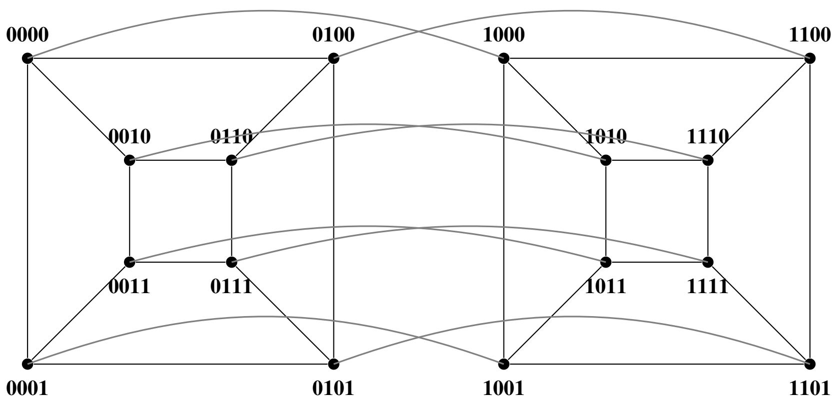

Figure 4: 4-dimensional hypercube Q_4

Below are distinct hyperplanes Q_3 obtained from Q_4 which form vertices of graph G.

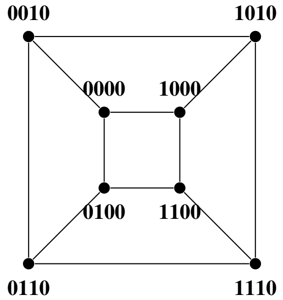

Figure 5: v_1

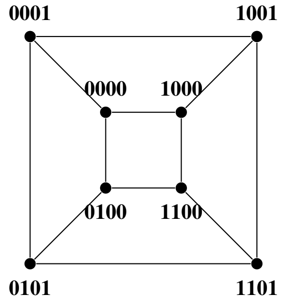

Figure 6: v_2

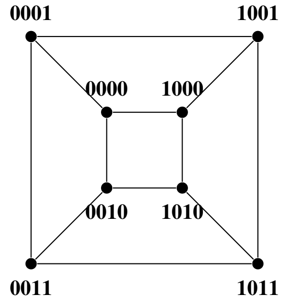

Figure 7: v_3

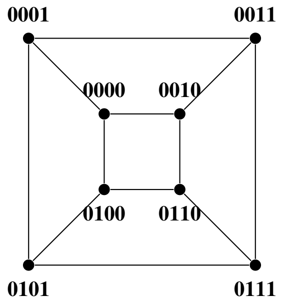

Figure 8: v_4

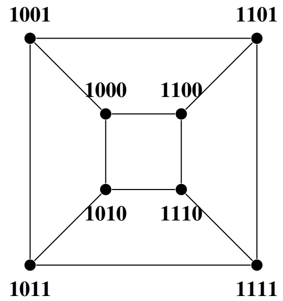

Figure 9: v_5

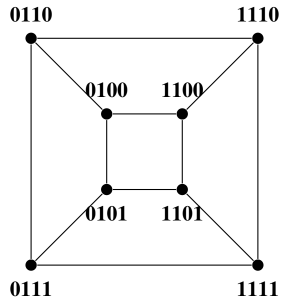

Figure 10: v_6

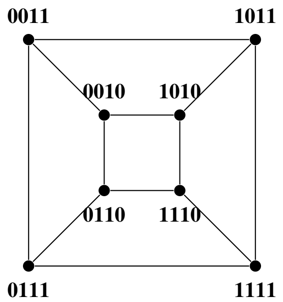

Figure 11: v_7

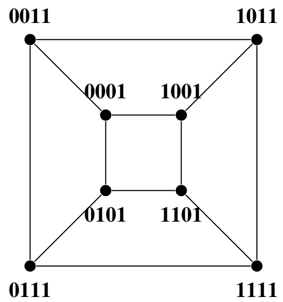

Figure 12: v_8

Considering these hyperplanes, Figure 5 - Figure 12, as vertices, we obtain adjacencies of each vertex as follows: $v_1 \rightarrow \{v_2, v_3, v_4, v_5, v_6, v_7\}$; $v_2 \rightarrow \{v_1, v_3, v_4, v_5, v_6, v_8\}$; $v_3 \rightarrow \{v_1, v_2, v_4, v_5, v_7, v_8\}$; $v_4 \rightarrow \{v_1, v_2, v_3, v_6, v_7, v_8\}$; $v_5 \rightarrow \{v_1, v_2, v_3, v_6, v_7, v_8\}$; $v_6 \rightarrow \{v_1, v_2, v_4, v_5, v_7, v_8\}$; $v_7 \rightarrow \{v_1, v_3, v_4, v_5, v_6, v_8\}$; $v_8 \rightarrow \{v_2, v_3, v_4, v_5, v_6, v_7\}$. We observe that each vertex v_i, for $1 \leq i \leq 8$, has regularity 6. Being unique eccentric vertex graph of regularity 6, any pair of adjacent vertices will have 4 common neighbours. Since G is unique eccentric vertex graph, each pair of eccentric vertices will have same set of neighbours. Hence, we see that $q = d$. Thus, graph G obtained is strongly regular with parameters (v', d, p, q) = $(8, 6, 4, 6)$.

4 Conclusion

Hypercube networks have always been an interesting topic for research enthusiasts in various fields due to their graph properties such as being bipartite, Hamiltonian, distance-regular, unique eccentric vertex graph and highly symmetric nature. In order to reduce the number of interconnections within hypercube network, without actually compromising on the robustness of a network, we have considered subgraphs which are also hypercubes of lower dimension, arising from hypercube Q_n that carry many important hereditary properties of the original network. Here, we obtained general expression for new class of PBIB-designs arising from hypercube Q_n where blocks are subgraphs Q_m of Q_n for $m < n$ and $n \geq 3$. These designs fall into the category of PBIB-designs with large number of associates classes which are found sparsely in literature. Also we give a construction of a strongly regular graph with $d = q$ arising from hypercubes Q_n taking vertices as hyperplanes.

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