TRIPLY-DIFFUSIVE CONVECTION IN COMPLETELY CONFINED FLUIDS

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ABSTRACT

The effect of uniform magnetic field on the triply-diffusive convection of an electrically conducting fluid completely confined in an arbitrary region bounded by rigid walls is considered. Some general qualitative results concerning the character of marginal state, stability of oscillatory motions and limitations on the oscillatory motions of growing amplitude, are derived. The results for the horizontal layer geometry in case of single-diffusive or double-diffusive or triply-diffusive fluids follow as a consequence.

Key Words: Triply-diffusive convection; Rayleigh numbers; Lewis numbers; Prandtl numbers; Chandrasekhar number.

1. Introduction

Thermohaline convection or more generally double diffusive convection has matured into a subject possessing fundamental departure from its counterpart, namely single diffusive convection, and is of direct relevance in the fields of oceanography, astrophysics, limnology and chemical engineering etc. For a broad view of the subject one may be referred to the review articles by Turner [1] and Brandt and Fernando [2]. An interesting early experimental study is that of Caldwell [3]. Two fundamental configurations have been studied in the context of thermohaline instability problem, the first one by Stern [4] wherein the temperature gradient is stabilizing and the concentration gradient is destabilizing and the second one by Veronis [5] wherein the gradient is destabilizing and the concentration gradient is stabilizing. The main results derived by Stern and Veronis for their respective configurations are that both allow the occurrence of a stationary pattern of motions or oscillatory motions of growing amplitude provided the destabilizing concentration gradient or the temperature gradient is sufficiently large. However, stationary pattern of motion is the preferred mode of setting in of instability in case of Stern’s configuration whereas oscillatory motions of growing amplitude are preferred in Veronis’ configuration. More complicated double-diffusive phenomenon appears if the destabilizing thermal/concentration gradient is opposed by the effects of a magnetic field or a rotation. In the domain of linear stability theory the double-diffusive convection problem can be described by a set of linear ordinary differential equations with constant coefficients and homogeneous boundary conditions. The task of finding the explicit analytical solutions of these equations (especially when boundaries are rigid) and thereby characterizing the critical conditions at the threshold of the instability are not entirely trivial since prohibitive amount of numerical work is required to affirm oscillatory or non-oscillatory motions as the eigen value equation involves all the parameters of the problem implicitly.

All the above researchers have considered the case of two component systems. However, it has been recognized later on by Griffiths [6], Turner [7] that there are many situations wherein more than two components are present. Examples of such multiple diffusive convection fluid systems include the solidification of molten alloys, geothermally heated lakes, and magmas and their laboratory models and sea water. Griffith [6], Pearlstein et al. [8] and Lopez [9] have theoretically studied the onset of convection in a horizontal layer, of infinite extension of a triply diffusive fluid (where the density depends on three independently diffusing agencies with different diffusivities).These researchers found that small concentrations of a third component with a smaller diffusivity can have a significant effect upon the nature of diffusive instabilities and oscillatory and direct salt finger modes are simultaneously unstable under a wide range of conditions, when the density gradients due to components with the greatest and smallest diffusivity are of same signs .Some fundamental differences between the double and triply convection are noticed by these researchers diffusive. Among these differences, one is that if the gradients of two of the stratifying agencies are held fixed ,then three critical values of the Rayleigh number of the third agency are sometimes required to specify the linear stability criteria(only one critical number is required in double diffusive convection) . Another difference is that the onset of convection may occur via a quassiperiodic bifurcation from the motionless basic state. Terrones [10] studied the effect of cross-diffusion on the stability criteria in a triply diffusive system. Ryzhkov and Shevtsova [11] studied the case of multicomponent mixture with application to thermogravitational column. Ryzhkov and Shevtsova [12] also studied the longwave instability of a multicomponent fluid with Soret effect. Rionero [13] studied a triply convective diffusive fluid mixture saturating a porous horizontal layer, heated from below and salted from above and obtained sufficient conditions for inhibiting the onset of convection and guaranteeing the global nonlinear stability of the thermal conduction solution. Rionero [14] also investigated the multicomponent diffusive convection in porous layer for the more general case when heated from below and salted by m salts partly from above and partly from below. Zhao, Wang and Zhang [15] investigated the problem of triply diffusive convection in Maxwell fluid saturated porous layer and obtained the criterion for the onset of stationary and oscillatory convection. Mohan and Kumar [16] investigated the problem of triply –diffusive magnetoconvection and established the relationship between various energies in Veronis type configurations. Mohan and Kumar [17] derived a semi-circle theorem that prescribed upper limits for the complex growth rate of oscillatory motions of neutral or growing amplitude in a triply-diffusive fluid layer.

Motivated by above considerations and the results of Veronis [5] and Stern [4] for the case of horizontal layer geometry with stress free boundaries and Sherman and Ostrach [18] for magnetohydrodynamic thermal stability problem in completely confined fluids, the present paper investigate the instability of triply-diffusive convection of a fluid completely confined in an arbitrary region bounded by rigid walls in the presence of of a uniform magnetic field applied in an arbitrary direction and derive some general qualitative results concerning the character of marginal state, stability of oscillatory motions and limitations on the oscillatory motions of growing amplitude. The results for the horizontal layer geometry in case if single-diffusive or double-diffusive or triply-diffusive fluids follow as a consequence.

2. Mathematical formulation and analysis

The relevant governing non-dimensional linearized perturbation equations in the present case with time dependence of the form exp (pt) ($p = p_r + ip_i$) are given by:

$$\frac{P}{\sigma}\vec{q} = -\nabla P - curlcurl\,\vec{q} + R_T\theta\hat{k} - R_S\phi_1\hat{k} - R_S{}'\phi_2\hat{k} + Q(curl\,\vec{h})\times\hat{l} \qquad (1)$$

$$\left(\nabla^2 - p\right)\theta = -\vec{q}\cdot\hat{k}\,, \qquad (2)$$

$$\left(\tau_1\nabla^2 - p\right)\phi_1 = -\vec{q}\cdot\hat{k}\,, \qquad (3)$$

$$\left(\tau_2\nabla^2 - p\right)\phi_2 = -\vec{q}\cdot\hat{k} \tag{4}$$

$$\text{curl curl } \vec{h} + \frac{p\sigma_1\vec{h}}{\sigma} = \text{curl}\left(\vec{q}\times\hat{\ell}\right) \quad , \tag{5}$$

$$\text{and} \qquad \nabla\cdot\vec{q} = 0 = \nabla\cdot\vec{h} \tag{6}$$

where $\vec{q}(x,y,z)$, $P(x,y,z)$, $\theta(x,y,z)$, $\phi_1(x,y,z)$, $\phi_2(x,y,z)$ and $\vec{h}(x,y,z)$ respectively denote the perturbed velocity, pressure, temperature, concentration of the first componenent, concentration of the second component and magnetic field and are complex valued functions defined on V, $R_T = \frac{g\alpha\beta d^4}{\kappa\upsilon}$ is the thermal Rayleigh number , $R_S = \frac{g\alpha_1\beta_1 d^4}{\kappa_1\upsilon}$ is the concentration Rayleigh number for the first componenent, $R_S{}' = \frac{g\alpha_2\beta_2 d^4}{\kappa_2\upsilon}$, $Q = \frac{\mu e H_0^2 d^2}{4\pi\rho_0\upsilon\eta}$ is the Chandrasekhar number, $\tau_1 = \frac{\kappa_1}{\kappa}$ and $\tau_2 = \frac{\kappa_2}{\kappa}$ are the Lewis numbers for the two concentration components with mass diffusivities κ_1 *and* κ_2 respectively and κ is the thermal diffusivity and $\hat{k}$ is a unit vertical vector. Further, with d as the characteristic length, the equations have been cast into dimensionless forms by using the scale factors $\frac{\kappa}{d}, \frac{d^2}{\kappa}, \beta d, \frac{\rho\upsilon\kappa}{d^2}, \beta_1 d\ , \beta_2 d$ *and* $\frac{\kappa H_0}{\eta}$ for velocity, time, temperature, pressure, two concentrations and magnetic field respectively.

Associated with the system of equations (1)-(6) is a set of homogeneous and time independent boundary conditions. We shall limit our consideration to the region V completely confined by rigid walls, which may be thermal, and concentration-wise conducting or insulating and to see the case when the electrical conductivity of the wall is large in comparison to the field (see Sherman and Ostrach [16]). Thus we seek solutions of equations (1)-(6) in the simply connected subset V of R^3 subject to the following boundary conditions:

$$\text{either } \vec{q} = 0 = \theta = \phi_1 = \phi_2 = \hat{n}\times curl\,\vec{h} \quad on \quad S \tag{7}$$

$$\text{or } \vec{q} = 0 = \nabla\theta\cdot\hat{n} = \nabla\phi.\hat{n} = \nabla\phi_2.\hat{n} = \hat{n}\times curl\,\vec{h} \quad on \quad S\,, \tag{8}$$

where $\hat{n}$ is a unit vector in the direction of the normal to boundary surface S.

Equations (1)-(6) together with boundary conditions (7)-(8) constitute an eigen-value problem for p for given values of other parameters and the system is stable, neutral or unstable according as pr is negative, p_r is zero or positive. Further,

(a) $p_i \neq 0$ and $p_r \geq 0$ describe oscillatory motion of neutral or growing amplitude.

(b) if $p_r = 0 \Rightarrow p_i = 0$, then the principle of exchange of stabilities (PES) is valid, otherwise, we have overstability.

We now, prove the following lemmas and theorems:

Lemma 1: (Poincre' inequality) – If f (x, y, z) is any smooth function which vanishes on S and ℓ is the smallest distance between two parallel planes which just contains V, then there exists a constant λ(> 2) such that

$$\int_V |\nabla f|^2 \, dv \geq \frac{\lambda}{\ell^2} \int |f|^2 \, dv \tag{9}$$

Proof: See Joseph [8].

Lemma 2 : If $\left(p, \vec{q}, \vec{h}, \theta, \phi_1, \phi_2\right)$ is a non-trivial solution of equation (1)-(6) together with either of the boundary conditions (7)-(8), then the following integral relations hold :

$$\int_V \vec{q}^* \cdot \text{curl curl}\, \vec{q} \, dv = \int_V |\text{curl } \vec{q}|^2 dv, \tag{10}$$

$$\int_V \vec{q}^* \cdot \text{curl curl}\left(\vec{q} \times \hat{\ell}\right) dv = \int_V \text{curl}\left(\vec{q} \times \hat{\ell}\right) \cdot \text{curl}\, \vec{q}^* \, dv, \tag{11}$$

$$\int_V \vec{q}^* \cdot curl\ curl\left(\theta \hat{k}\right) dv = 0 = \int_V \vec{q}^* \cdot curl\ curl\left(\phi_1 \hat{k}\right) dv$$
$$= \int_V \vec{q}^* \cdot curl\ curl\left(\phi_2 \hat{k}\right) dv, \tag{12}$$

$$\int_V \vec{q}^* \cdot \left[\left(\text{curl}\, \vec{h}\right) \times \hat{\ell}\right] dv = -\int_V \vec{h} \cdot \text{curl curl}\left(\vec{q}^* \hat{\ell}\right) dv, \tag{13}$$

$$\int_V \vec{q}^* \cdot \left[\hat{\ell} \text{ curl curl curl } \vec{h}\right] dv = -\int_V \text{curl curl } \vec{h} \cdot \text{ curl}\left(\vec{q}^* \hat{\ell}\right) dv, \tag{14}$$

$$\int_V \vec{h}^* \cdot \text{curl curl curl } \vec{h} \, dv = -\int_V \left|\text{ curl } \vec{h}\right|^2 dv = \int_V \vec{h}^* \cdot \text{curl curl curl } \vec{h}^* \, dv, \tag{15}$$

$$\int_{V} \vec{q}^{\,*} \cdot \nabla(P)\,dv = 0\,, \tag{16}$$

$$\int_{V} \vec{q}^{\,*} \cdot \left[\nabla\left(div\,\theta\hat{k}\right)\right] dv = 0 = \int_{V} \vec{q}^{\,*} \cdot \left[\nabla\left(div\phi_1\hat{k}\right)\right] dv$$
$$= \int_{V} \vec{q}^{\,*} \cdot \left[\nabla\left(div\phi_2\hat{k}\right)\right] dv \tag{17}$$

$$\int_{V} \vec{q}^{\,*} \cdot \left[\nabla\left(\hat{\ell} \cdot \text{curl curl } \vec{h}\right)\right] dv = 0\,, \tag{18}$$

$$\int_{V} \theta^* \nabla^2 \theta dv = -\int_{V} \left|\nabla\theta\right|^2 dv = \int \theta^* \nabla \theta^* dv\,, \tag{19}$$

$$\int_{V} \phi_1^* \nabla^2 \phi_1 dv = -\int \left|\nabla\phi_1\right|^2 dv = \int \phi_1^* \nabla \phi_1^* dv\,,$$

and (20)

$$\int_{V} \phi_2^* \nabla^2 \phi_2 dv = -\int \left|\nabla\phi_2\right|^2 dv = \int \phi_2^* \nabla \phi_2^* dv$$

where '*' denotes complex conjugate and $\left|\vec{A}\right|^2 = \vec{A} \cdot \vec{A}^*$ for any vector $\vec{A}$.

Proof: If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are smooth vector-valued functions and ψ is a smooth scalar-valued function on V such that $\vec{A} \times \vec{B}$ and $\Psi\vec{C}$ vanish on S, then using Gauss' divergence theorem and the vector identities

$$\text{div}\left(\vec{A} \times \vec{B}\right) = \vec{B} \bullet \text{curl } \vec{A} - \vec{A} \quad \text{curl}\,\vec{B}$$

and

$$div\left(\Psi\vec{C}\right) = \nabla\Psi.\vec{C} + \Psi\ div\ \vec{C}$$

it follows that

$$\int_{V} \vec{B} \cdot \text{curl } \vec{A}\ dv = \int_{V} \vec{A} \cdot \text{curl } \vec{B}\ dv\,, \tag{21}$$

and

$$\int_{V} \nabla\Psi \cdot \vec{C}\, dv = -\int_{V} \Psi\, div\, \vec{C}\, dv \quad . \tag{22}$$

Now integral relations (10) – (15) follow from equation (21) by choosing $\vec{A}$ and $\vec{B}$ appropriately and integral relations (16)-(20) follow from equation (22) by choosing Ψ and $\vec{C}$ appropriately.

This completes the proof of the lemma.

Theorem 1 : If $\left(p, \vec{q}, \vec{h}, \theta, \phi_1, \phi_2\right)$, p = pr +ipi, is a non-trivial solution of equations(1) - (6) together with either of the boundary conditions (7) - (8), $R_T > 0$, $R_S > 0$, $R_S' > 0$ and $R_T < \min\left\{\frac{2R_S}{\tau_1}, \frac{2R_S'}{\tau_2}\right\}$, $\tau_2 R_S = \tau_1 R_S'$, then

$p_r = 0 \Rightarrow p_i \neq 0$

Proof : Suppose $p_r = 0 \Rightarrow p_i = 0$ then p = 0, and equations (1)-(5) become

$$\nabla P + curl\ curl\, \vec{q} = R_T\theta\, \hat{k} - R_S\phi_1\hat{k} - R_S'\phi_2\hat{k} + Q\left(curl\ \vec{h}\right)\times \ell, \tag{23}$$

$$\nabla^2\theta = -\vec{q}\cdot\hat{k}, \tag{24}$$

$$\nabla^2\left(\tau_1\phi_1\right) = -\vec{q}\cdot\hat{k}, \tag{25}$$

$$\nabla^2\left(\tau_2\phi_2\right) = -\vec{q}\cdot\hat{k}, \tag{26}$$

$$\text{curl curl}\,\vec{h} = \text{curl}\left(\vec{q}\cdot\hat{\ell}\right). \tag{27}$$

If $\zeta = \theta - \frac{1}{2}\tau_1\phi_1 - \frac{1}{2}\tau_2\phi_2$ then it follows from equations (24)-(26)

$$\nabla^2\zeta = 0 \tag{28}$$

Further, in view of boundary conditions (7)-(8), we have either

$$\zeta = 0 \ \text{ or } \ \nabla\zeta\cdot\hat{n} = 0 \ \text{ on } S \tag{29}$$

The only solution of equation (28) in V subject to either of the boundary condition in (29) is $\zeta = 0$. Consequently equation (23) assume the form

$$\nabla P + curl\ curl\,\vec{q} = \left\{\frac{\tau_1 R_T}{2} - R_S\right\}\phi_1\hat{k} + \left\{\frac{\tau_2 R_T}{2} - R_S'\right\}\phi_2\hat{k} + Q\left(curl\ \vec{h}\right)\times\hat{\ell} \quad (30)$$

Taking dot product of equation (30) with $\vec{q}^*$, integrating the resulting equation over the domain V and using lemma 2, we get

$$\int_V \left|curl\,\vec{q}\right|^2 dv + Q\int \vec{h}\cdot curl\left(\vec{q}\times\hat{\ell}\right)dv = \left(\frac{\tau_1 R_T}{2} - R_S\right)\int_V \phi_1 (q^*.\hat{k})dv$$

$$+\left(\frac{\tau_2 R_T}{2} - R_S'\right)\int_V \phi_2 (q^*.\hat{k})dv \quad (31)$$

Equation (31) upon using equation (25) (26) and (27) and then appealing to lemma 2 yields the equation

$$\int_V \left|curl\,\vec{q}\right|^2 dv + Q\int_V \left|curl\,\vec{h}\right|^2 = \tau_1\left(\frac{\tau_1 R_T}{2} - R_S\right)\int_V \left|\nabla\phi_1\right|^2 dv + \tau_2\left(\frac{\tau_2 R_T}{2} - R_S'\right)\int_V \left|\nabla\phi_2\right|^2 dv\,. \quad (32)$$

It follows from equation (32) that

$\frac{\tau_1 R_T}{2} > R_S$ and $\frac{\tau_2 R_T}{2} > R_S'$

i.e. $R_T > \max\left\{\frac{2R_S}{\tau_1}, \frac{2R_S'}{\tau_2}\right\}$

a result contrary to the given hypothesis of the theorem. Hence $p_r = 0 \Rightarrow p_i \neq 0$. This completes the proof of the theorem.

Theorem 1, in the parlance of linear stability theory, may be stated as follows:

> PES is not valid for the hydromagnetic triple-diffusive convection in completely confined fluids if
>
> $$R_T < \min\left\{\frac{2R_S}{\tau_1}, \frac{2R_S'}{\tau_2}\right\},\ \tau_2 R_S = \tau_1 R_S'\,.$$

The following corollaries are direct consequences of Theorem 1

Cor.1 PES is not valid for hydromagnetic triple-diffusive convection in completely confined fluids if $R_T < \min\left\{\frac{2R_S}{\tau_1}, \frac{2R_S'}{\tau_2}\right\}, \tau_2 R_S = \tau_1 R_S'$.

Cor. 2 – PES is not valid for hydromagnetic double-diffusive convection of Veronis type in completely confined fluids if $\frac{\tau_1}{2} R_T \leq R_S$ (R_S'=0).

Theorem 2 : If $\left(p, \vec{q}, \vec{h}, \theta, \phi_1, \phi_2\right)$, p = pr +ipi, is a non-trivial solution of equations(1) - (6) together with either of the boundary conditions (7) - (8), $R_T > 0$, $R_S > 0$, $R_S' > 0$ and $R_T < \min\left\{\frac{2R_S}{\tau_1}, \frac{2R_S'}{\tau_2}\right\}, \tau_2 R_S = \tau_1 R_S', \tau_1 = \tau_2 = 1$, then

$p_r < 0$

Proof : Since $\tau_1 = \tau_2 = 1$, therefore it follows from equations (2)-(4)and boundary conditions(7)-(8) that

$$\nabla^2 \theta = -\vec{q} \cdot \hat{k}, \tag{24}$$

$$\nabla^2 \left(\tau_1 \phi_1\right) = -\vec{q} \cdot \hat{k}, \tag{25}$$

$$\nabla^2 \left(\tau_2 \phi_2\right) = -\vec{q} \cdot \hat{k}, \tag{26}$$

If $\zeta = \theta - \frac{1}{2}\tau_1 \phi_1 - \frac{1}{2}\tau_2 \phi_2$ then it follows from equations (24)-(26)

$$\nabla^2 \zeta = 0 \tag{28}$$

Further, in view of boundary conditions (7)-(8), we have either

$$\zeta = 0 \text{ or } \nabla\zeta \cdot \hat{n} = 0 \text{ on S} \tag{29}$$

The only solution of equation (28) in V subject to either of the boundary condition in (29) is ζ= 0. Consequently equation (23) assume the form

$$\nabla P + curl\ curl\, \vec{q} = \left\{\frac{\tau_1 R_T}{2} - R_S\right\}\phi_1 \hat{k} + \left\{\frac{\tau_2 R_T}{2} - R_S'\right\}\phi_2 \hat{k} + Q\left(curl\ \vec{h}\right) \times \hat{\ell} \tag{30}$$

Taking dot product of equation (30) with $\vec{q}^*$, integrating the resulting equation over the domain V and using lemma 2, we get

$$\int_{V} |curl\,\vec{q}|^2\, dv + Q\int \vec{h}\cdot curl\left(\vec{q}\times\hat{\ell}\right)dv = \left(\frac{\tau_1 R_T}{2} - R_S\right)\int_V \phi_1 (q * .\hat{k})dv$$

$$+\left(\frac{\tau_2 R_T}{2} - R_S{}'\right)\int_V \phi_2 (q * .\hat{k})dv \qquad (31)$$

Equation (31) upon using equation (25) (26) and (27) and then appealing to lemma 2 yields the equation

$$\int_{V} |curl\,\vec{q}|^2\, dv + Q\int_{V} \left|curl\,\vec{h}\right|^2 = \tau_1\left(\frac{\tau_1 R_T}{2} - R_S\right)\int_V |\nabla\phi_1|^2 dv + \tau_2\left(\frac{\tau_2 R_T}{2} - R_S{}'\right)\int_V |\nabla\phi_2|^2 dv\,. \qquad (32)$$

It follows from equation (32) that

$$\frac{\tau_1 R_T}{2} > R_S \quad \text{and} \quad \frac{\tau_2 R_T}{2} > R_S{}'$$

i.e. $R_T > \max\left\{\frac{2R_S}{\tau_1}, \frac{2R_S{}'}{\tau_2}\right\}$

a result contrary to the given hypothesis of the theorem. Hence $p_r = 0 \Rightarrow p_i \neq 0$. This completes the proof of the theorem.

Theorem 1, in the parlance of linear stability theory, may be stated as follows:

PES is not valid for the hydromagnetic triple-diffusive convection in completely confined fluids if

$$R_T < \min\left\{\frac{2R_S}{\tau_1}, \frac{2R_S{}'}{\tau_2}\right\},\ \tau_2 R_S = \tau_1 R_S{}'\,.$$

The following corollaries are direct consequences of Theorem 1

Cor.1 PES is not valid for hydromagnetic triple-diffusive convection in completely confined fluids if $R_T < \min\left\{\frac{2R_S}{\tau_1}, \frac{2R_S{}'}{\tau_2}\right\},\ \tau_2 R_S = \tau_1 R_S{}'$.

Cor. 2 – PES is not valid for hydromagnetic double-diffusive convection of Veronis type in completely confined fluids if $\frac{\tau_1}{2} R_T \le R_S$ ($R_S{}'$ =0).

THEOREM 3 – If $\left(p, \vec{q}, \vec{h}, \theta, \phi_1, \phi_2\right)$, p = pr +ipi, is a non trivial solution of equations (1) – (5) with either of the boundary conditions (6) – (7), $R_T > 0$, $R_S > 0$, $R_S' > 0$ and $\tau < \delta \le 1$, then for large Q (or for large $|R_S|$ if Q = 0)

$$pr \ge 0 \Rightarrow pi = 0,$$

where $\delta = \begin{cases} 1, \text{ if } R_S < 0, \text{ and } Q = 0 \\ \frac{\sigma}{\sigma_1}, \text{ if } R_S \le 0 \text{ and } Q > 0. \end{cases}$

PROOF : Operating on equation (1) by (δ curl curl +p) and using the vector identities

$$\text{curl}\left(\Psi\vec{A}\right) = \Psi \text{curl}\,\vec{A} + \nabla\Psi + \vec{A}$$

$$\text{curl}\left(\vec{A}\times\vec{B}\right) = \left(\vec{B}\cdot\nabla\right)\vec{A} - \left(\vec{A}\cdot\nabla\right)\vec{B} + \vec{A}\text{ div }\vec{B}$$

and

$$\nabla\left(\vec{A}\times\vec{B}\right) = \left(\vec{B}\cdot\nabla\right)\vec{A} + \left(\vec{A}\cdot\nabla\right)\vec{B} + \vec{B}\text{ curl }\vec{A} + \vec{A}\times\text{curl }\vec{B},$$

with an appropriate choice of $\Psi, \vec{A}$ and $\vec{B}$, it follows that

$$\begin{aligned} & p\left(1+\frac{\delta}{\sigma}\right) curl\ curl\ \vec{q} + \frac{p^2}{\sigma}\vec{q} + p\nabla P \\ & \quad -|R_S'|\left\{\delta\left[\nabla\left(\text{div }\phi\hat{\beta}\right) - \nabla^2\phi\hat{\beta}\right] + p\phi\hat{\beta}\right\} \\ & \quad -|R_T'|\left\{\delta\left[\nabla\left(\text{div }\theta\hat{\beta}\right) - \nabla^2\theta\hat{\beta}\right]\right\} + \\ & \quad + Q\left\{\delta\left[\hat{\ell}\times\text{curl curl curl }\vec{h} - \nabla\left(\hat{\ell}\times\text{curl curl }\vec{h}\right) - p\left(\text{curl }\vec{h}\right)\times\hat{\ell}\right]\right\} \\ & \quad = -\delta\text{ curl curl curl curl }\vec{q} \end{aligned} \tag{39}$$

Taking the dot product of equation (39) with $\vec{q}^*$, integrating the resulting equation over the domain V and lemma 2, we have

$$p\left(1+\frac{\delta}{\sigma}\right)\int_{v}|\text{curl }\vec{q}|^{2}+\frac{p^{2}}{\sigma}\int_{v}|\vec{q}|^{2}dv-|R'_{T}|\int_{v}\left(\delta\nabla^{2}\theta-p\theta\right)\left(\vec{q}^{*}\cdot\hat{\beta}\right)dv$$

$$+|R'_{S}|\int_{v}\left(\delta\nabla^{2}\phi-p\phi\right)\left(\vec{q}^{*}\cdot\hat{\beta}\right)dv+pQ\int_{v}\vec{h}\text{ curl}\left(\vec{q}\cdot\hat{\ell}\right)dv$$

$$+Q\delta\int_{v}\text{curl curl }\vec{h}\text{ curl}\left(\vec{q}\cdot\hat{\ell}\right)dv$$

$$=-\delta\int_{v}\vec{q}\ curl\ curl\ curl\ curl\ \vec{q}\ dv \tag{40}$$

Since Q (the ratio of magnetic to viscous forces) is very large, the effect of viscosity is thus significant near the bounding surfaces and in the above equation the integral on the right hand side (resulting from the viscous forces) is negligible in comparison with the last integral on the left hand side (resulting from the magnetic force) (C.F. Sherman and Ostrach). Consequently, taking the right hand side of equation (40) to zero, eliminating $\left(\vec{q}^{*}\beta^{*}\right)$ and $\left(\vec{q}^{*}\times\ell^{*}\right)$ from the resulting equation by using equation (33)-(35) and then appealing to lemma 2, we get

$$p\left(1+\frac{\delta}{\sigma}\right)\int_{v}|\text{curl }\vec{q}|^{2}dv+\frac{p^{2}}{\sigma}\int_{v}|\vec{q}|^{2}dv+|R'_{T}|\int_{v}\left(\delta\left(\nabla^{2}\theta\right)^{2}+|p|^{2}|\theta|^{2}\right)dv+|R'_{T}|\left(p^{*}\delta+p\right)$$

$$\int_{v}|\nabla\theta|^{2}dv-|R'_{S}|\int_{v}\delta\left(\left|\nabla^{2}\theta\right|^{2}+|p|^{2}|\phi|^{2}\right)dv-|R'_{S}|\left(p^{*}\delta+\tau p\right)\int_{v}|\nabla\phi|^{2}dv$$

$$+Q\int_{v}\left[\delta\left|\text{curl curl }\vec{h}\right|^{2}+\frac{|p|^{2}\sigma_{1}}{\sigma}\left|\vec{h}\right|^{2}\right]dv$$

$$+Q\left[\frac{p^{*}\delta\sigma_{1}}{\sigma}+p\right]\int_{v}\left|\text{curl}\vec{h}\right|^{2}dv=0 \tag{41}$$

Equating the imaginary past of equation (41) to zero and assuming pi ≠ 0, we get

$$\left(1+\frac{\delta}{\sigma}\right)\int_{v}|\text{curl }\vec{q}|^{2}+\frac{2pr}{\sigma}\int_{v}|\vec{q}|^{2}dv+|R'_{T}|(1-\delta)\int_{v}|\nabla\theta|^{2}dv$$

$$+|R'_{S}|(\delta-\tau)\int_{v}|\nabla\phi|^{2}dv=0 \tag{42}$$

Equation (42) cannot obviously be satisfied under the conditions of the theorem. Hencewe must have

$$p_{i}=0$$

This comples the proof of the theorem.

Theorem 2 implies that the hydromagnetic GSTHC on arbitrary neutral or unstable mode is definitely non-oscillatory in character and in particular PES is valid if $\tau\sigma_1 < \sigma \leq \sigma_1$. Further, this theorem also implies the validity of this result for the GSTHC if $\tau < 1$.

THEOREM 3 : If $(p, \vec{q}, \theta, \phi, \vec{h})$, $p = p_r + ip_i$, $p_r \geq 0$, $p_i \neq 0$ is a non-trivial solution of equation (1)-(5) together with the boundary conditions (6) and $R_T < 0, R_S < 0$ and $\delta > 1$, then for large Q (or for large $|R_S|$ if Q = 0)

$$|p| < \hat{\delta}\left[|R'_T|(\delta-1)B^2 + |R'_S|\right],$$

where

$\hat{\delta} = \dfrac{\ell^2\sigma}{\lambda(\sigma+\delta)}$, δ is as in Theorem 2 and ℓ and λ are as in Lemma 1.

PROOF : It follows from equation (23) that

$$\int_V (\nabla^2\theta - \rho\theta)(\nabla^2\theta^* - \rho^*\theta^*)dv = B^2 \int_V |\vec{q}\cdot\hat{\beta}|^2 dv \quad (43)$$

Equation (43) upon using lemma2 gives

$$\int_V |\nabla^2\theta|^2 dv + 2pr\int_V |\nabla\theta|^2 dv + |p|^2 \int_V |\theta|^2 dv = \int_V |\vec{q}\cdot\hat{\beta}|^2 dv \quad (44)$$

Equation (44), upon using pr ≥ 0, pi ≠ 0 give

$$\int_V |\theta|^2 dv < \frac{B^2}{|p|^2}\int_V |\vec{q}\cdot\hat{\beta}|^2 dv \leq \frac{B^2}{|p|^2}\int_V |\vec{q}|^2 dv \quad (45)$$

Again multiplying (33) by θ*, integrating over the domain V, using lemma 2 and equating the real parts of the resulting equation, we have

$$\int_V |\nabla\theta|^2 dv + pr\int_V |\theta|^2 dv = \text{Real part of}\left(B\cdot\int_V \theta^* |\vec{q}\cdot\hat{\beta}| dv\right)$$

$$\leq \left|B\cdot\int_V \theta^*(\vec{q}\cdot\hat{\beta})dv\right| \leq B\cdot\int_V |\theta|\ |\vec{q}\cdot\hat{\beta}| dv$$

which upon using schwartz's inequality and the fact that pr ≥ 0, gives

$$\int_{v}|\nabla\theta|^{2}dv \le B\cdot\left[\int_{v}|\theta|^{2}dv\right]^{1/2}\left[\left(B\cdot\int_{v}|\vec{q}\cdot\hat{k}|^{2}dv\right)\right]^{1/2}$$

$$\le B\cdot\left[\int_{v}|\theta|^{2}dv\right]^{1/2}\left[\left(B\cdot\int_{v}|\vec{q}|^{2}dv\right)\right]^{1/2}. \qquad (46)$$

Combining inequalities (45) and (46), we get

$$\int_{v}|\nabla\theta|^{2}dv < \frac{B\cdot B}{|p|}\int_{v}|\vec{q}|^{2}dv. \qquad (47)$$

Further, the solenoidal character of the velocity field $\vec{q}$ namely div $\vec{q}=0$, implies that

$$\int_{v}|\text{curl }\vec{q}|^{2}dv = \int_{v}\left(\vec{q}^{*}\cdot\text{curl curl }\vec{q}\right)dv = -\int_{v}\vec{q}^{*}\nabla^{2}\vec{q}dv$$

which upon taking $\vec{q}=(u,v,w)$, gives

$$\int_{v}|\text{curl }\vec{q}|^{2}dv = \int_{v}\left(|\nabla U|^{2}+|\nabla V|^{2}+|\nabla W|^{2}\right)dv \qquad (48)$$

Equation (48) together with lemma 1 yields the inequality

$$\int_{v}|\vec{q}|\ dv < \frac{\ell^{2}}{\lambda}\int_{v}|\text{curl }\vec{q}|^{2}dv \qquad (49)$$

Inequality (47) and (49) implies that

$$\int_{v}|\nabla\theta|^{2}dv < \frac{B^{2}\ell^{2}}{\lambda|p|}\int_{v}|\text{curl }\vec{q}|^{2}dv \qquad (50)$$

Similarly proceeding from equation (34), and emulating the steps in the derivation of inequality (50), we have

$$\int_{v}|\nabla\phi|^{2}dv < \frac{\ell^{2}}{\lambda\tau|p|}\int_{v}|\text{curl }\vec{q}|^{2}dv \qquad (51)$$

Using inequality (5) and (51) in equation (42), we get

$$\left(\frac{\sigma+\delta}{\sigma}\right)\left\{|p|-\hat{\delta}\left\langle|R'_{T}|(\delta-1)B^{2}+|R'_{S}|\right\rangle\right\}\int_{v}|\text{curl }\vec{q}|^{2}dv$$

$$+\frac{2pe}{\sigma}\int_{v}|\vec{q}|^{2}dv+\delta|R'_{S}|\int_{v}|\nabla\phi|^{2}dv<0, \qquad (52)$$

Inequality (52) clearly implies that

$$|p| \langle \hat{\delta} \left[|R'_T| (\delta - 1) B^2 + |R'_S| \right].$$

This completes the proof of the theorem.

Theorem 3 implies that the complex growth rate of an arbitrary oscillatory perturbation which may be neutral or unstable for the hydromagnetic GSTHC lies inside a semi-circle with centre origin and

$$\text{Radius}= \hat{\delta} \left[|R'_T| (\delta - 1) B^2 + |R'_S| \right], \left(\delta = \frac{\sigma}{\sigma_1} \right)$$

in the right half of the complex p-plane.

3. Conclusion

The present paper investigates the instability of Dufour-driven thermosolutal convection of a fluid completely confined in an arbitrary region bounded by rigid walls in the presence of a uniform magnetic field applied in an arbitrary direction. It has been found that Principle of exchange of stabilities is not valid for the hydromagnetic generalized Stern's thermohaline configuration if $|R_S| \leq \tau |R_T| \left\langle 1 - \frac{R_3 \gamma}{\tau} \right\rangle$. Secondly, for large Chandrasekhar number and $\tau < \delta < 1$, a neutral or unstable mode is definitely non oscillatory in character and in particular PES is valid. Finally the complex growth rate of an arbitrary oscillatory perturbation which may be neutral or unstable lies inside a semi-circle with centre origin and radius = $\hat{\delta} \left[|R'_T| (\delta - 1) B^2 + |R'_S| \right]$ in the right half of the complex p-plane. Further, the results for the horizontal layer geometry in case of single-diffusive or double diffusive fluids follows as a consequence by taking $\lambda = \pi^2, l = 1$ respectively.

Nomenclature

p = growth rate, $\left[\frac{1}{s} \right]$

$\vec{q}$ = Velocity, $\left[\frac{m}{s} \right]$

σ = Prandtl number, $\left[\frac{\nu}{\kappa} \right]$, $[-]$

P = Pressure, $[Pa]$

R_T = Thermal Rayleigh number, $[-]$

R_S = Solutal Rayleigh number, $[-]$

Q = Chandrasekhar number, $[-]$

$\vec{h}$ = Magnetic field, [Gauss]

R_3 = Gradient ratio, $\left[\frac{\beta'}{\beta}\right]$

γ = Dufour number, $[-]$

τ = Lewis number, $\left[\frac{\kappa'}{\kappa}\right]$

σ_1 = Magnetic Prandtl number $\left[\frac{\nu}{\eta}\right]$, $[-]$

g = acceleration due to gravity, $\left[\frac{m}{s^2}\right]$

d = depth of layer, $[m]$

t = time, $[s]$

Greek letter

α = coefficient of thermal expansion, $\left[\frac{1}{K}\right]$

α' = coefficient of solute expansion

β = uniform temperature gradient, $\left[\frac{K}{m}\right]$

β' = uniform concentration gradient, $\left[\frac{K}{m}\right]$

η = electrical resistivity, $\left[\frac{m^2}{s}\right]$

κ = thermal diffusivity, $\left[\frac{m^2}{s}\right]$

κ' = mass diffusivity,

ν = kinematic viscosity, $\left[\frac{m^2}{s}\right]$

ρ = density, $\left[\frac{kg}{m^3}\right]$

θ = perturbation in temperature, $[K]$

ϕ = perturbation in concentration, $[Kg]$

δ = dimensionless ratio of two Prandtl numbers, $[-]$

γ = dimensionless Dufour number, $[-]$

Γ = dimensionless ratio of two Rayleigh numbers, $[-]$

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